ELASTICITY

Syllabus:

Module-2 : Elasticity

Elasticity: Concept of elasticity, plasticity, stress, strain, tensile stress, shear stress, compressive stress, strain hardening and strain softening, failure (fracture/fatigue). Hooke's law, different elastic moduli: Poisson's ratio, Expression for Young's modulus (Y), Bulk modulus (K) and Rigidity modulus (n) in terms of α and β. Relation between Y, n and K, Limits of Poisson's ratio.

Bending of beams: Neutral surface and neutral plane, Derivation of expression for bending moment. Bending moment of a beam with circular and rectangular cross section. Single cantilever, derivation of expression for Young's' modulus.

Torsion of cylinder: Expression for couple per unit twist of a solid cylinder (Derivation).
Torsional pendulum- Expression for period of oscillation.

Elastic Properties of Materials

The property of a body by virtue of which it regains its original shape or size on removal of external deforming force is known as elasticity.

Elastic bodies are those bodies which regains its original shape or size after the removal of deforming force. Deforming force is external force which always tends to change the size or shape of the rigid body. A rigid body is material having definite shape and volume.
Ex: Steel, Brass, etc.

The concept of electricity can be understood using spring-ball model. In solids atoms or molecules experience intermolecular forces. The intermolecular force between the molecules in a solid is maximum, if each molecule of a solid is in its equilibrium position. If the position of molecules is changed due to application of external force, then intermolecular force between them also changes. This disturbed molecule always has a tendency to go back to the mean position. This behavior of molecules leads to elasticity. For simplicity we imagine two molecules are connected by a spring. A restoring force is developed in a spring when the molecules or atoms are disturbed from the mean position.

The maximum value of deforming force up to which the body regains its original size or shape is called elastic limit. If the applied force exceeds the elastic limit then the body does not regains its original size or shape and becomes non elastic or plastic body.

Perfectly elastic body: Are those bodies which completely regain its original size or shape. Practically, perfectly elastic body is ideal in nature, because nobody can regain 100% of its original size or shape.

Plastic or non-elastic body: Are those bodies which do not regain its original size or shape after the removal of deforming force.
Ex: wet clay, putty, wax, etc.
The concept of plasticity is arises because of shifting of molecules permanently due to the application of external force.

All the plastic materials using into this world are synthetic they are classified in to two types

(a) **Thermo plastics:** they are polymers containing long chain molecules and they can be melted many times and recast again and again, i.e. they are reversible.
Ex: Nylon, Polystyrene, etc.

(b) **Thermo setting plastics:** these plastics are irreversible; they can be melted and molded only one time. Any attempt to melt them leads to cracking or disintegration.
Ex: Bekalite, Resin, etc,

Importance of elastic materials in engineering:

Elastic bodies are of great importance to us as they are used in construction of building using steel or Iron in making machines and machinery parts and also engines in automobiles, I-shaped girders, cables, ropes, artificial limbs, electric poles etc. When a tool is made up of Iron is used in an application where there is a lot of vibration then a small crack or fracture leads to damage the tool. In order to overcome, these tools are made of steel which is more elastic than iron and can withstand vibration and does not break easily.

Generally pure metals are not used for engineering applications because they have low strength and ductile. Hence alloys are used instead of pure metals because of unique properties of alloys.

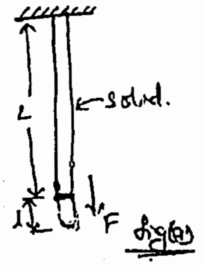
Concept of stress and strain:

Stress: The restoring force developed per unit area of body is called stress.
Restoring force is equal and opposite to the applied force or deforming force.

$$\therefore \text{stress} = \frac{\textbf{Restoring force}}{\textbf{unit area of the body}} = \frac{F}{A}$$

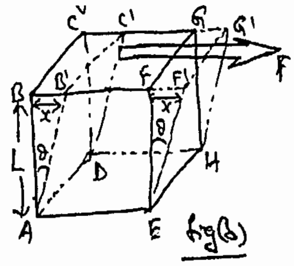
SI unit of stress is Nm^{-2}(Newton per meter square)

Depends on the nature of external applied force, stress is classified in to three type, they are



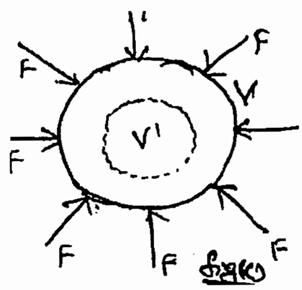
1. **Tensile Stress or Longitudinal stress:** It is defined as the restoring force acting per unit area perpendicular to the cross section of body.
   or
   It is the stretching force acting per unit area of the section of the solid along its length.

$$\therefore \textbf{Longitudinal or Tensile Stress} = \frac{F}{A}$$



2. **Tangential Stress or shear stress:** It is defined as the ratio of the restoring force acting tangent to the surface area of body.
   or
   It is the force acting tangentially per unit area on the surface of the body.

$$\therefore \textbf{Shear stress} = \frac{\textbf{tangential force}}{\textbf{unit area of the body}} = \frac{F}{A}$$



3. **Volume Stress or compressive stress:** It is defined as the ratio of the restoring force acting all over the body per unit area.
   or
   It is the uniform pressure (force per unit area) acting normal all over the body. If the force acts inward, there is a compression. If the force acts outward then there is a expansion of the body.
   In figure, F s the inward force, V is the original volume and V^1 is the compressed volume.

Strain: It is defined as ratio of change in dimension to the original dimension

$$\therefore \text{stress} = \frac{\textbf{change in dimension}}{\textbf{original dimension}}$$

Strain has no unit and it indicates deformation of dimension of body. Based on nature of the deformation strain is classified in to three types.

1. **Linear or Longitudinal strain:** It is defined as ratio of change in length to the original length of the body.

$$\therefore \textbf{Linear or Longitudinal strain} = \frac{\textbf{change in length}}{\textbf{original length}} = \frac{\textbf{l}}{\textbf{L}}$$

2. **Shear strain:** It is defined as the ratio of shift or displacement of height of a body to the original height when tangential forces are applied. Shear strain itself is a measure of shearing angle (**θ**).
   It is the force acting tangentially per unit area on the surface of the body.

$$\therefore \textbf{Shear strain} = \textbf{Shear angle}(\boldsymbol{\theta}) = \frac{\textbf{BB}^1}{\textbf{AB}}$$

$$\therefore \boldsymbol{\theta} = \frac{\textbf{x}}{\textbf{L}}$$

3. **Volume Strain:** It is defined as change in volume to the original volume of the body

$$\therefore \textbf{Volume strain} = \frac{\textbf{change in volume}}{\textbf{original volume}} = \frac{\textbf{v}^1}{\textbf{V}}$$

Factors affecting Elasticity:

1. **Stress:** when certain materials are subjected to continuous stress, then the material undergo deformation continuously. This is known as creep. The body shows slow plastic deformation and cannot regain its complete original size and experience permanent set which leads to fracture.
   Ex: when the heavy object kept on the wooden shelf for a long time, shape of the wooden shelf deformed permanently.
2. **Temperature:** When the body is subjected to high temperature, the effect of creep is observed. This creep decreases the strength of the body and leads to fracture. While designing boilers, turbines, jet engines, etc. high temperature is observed which always decreases elasticity.
3. **Annealing:** Annealing is a process of heating the material and cooling slowly or gradually. Annealing improves the elasticity and increases ductility of material due to which strength and toughness is enhanced to meet the requirement of machinery parts.
4. **Impurity:** Elasticity of a material can be increased or decreased by adding impurities.
   Some type of impurities obstructs the motion of dislocations in the lattice hence the elasticity increases and also yields strength.
   Some type of impurities allows or transfers the motion of dislocations and leads to crack or fracture. Hence elasticity decreases due to low strength.

Hooke's Law:

Statement: For a small strain, stress is directly proportional to strain
Or the stress is directly proportional to strain within elastic limit.

⇒ Stress α Strain

Stress = constant x Strain

$$\Rightarrow \quad \frac{\textbf{Stress}}{\textbf{Strain}} = \textbf{constant} = \text{Modulus of elasticity}$$

The constant is known as modulus of elasticity and it is defined as the ratio of stress to strain. SI unit of modulus of elasticity is Nm^{-2}

There are 3 types of modulus of elasticity (Elastic Moduli)

(1) **Young's modulus (Y):** it is defined as the ratio of longitudinal stress to linear strain, with in elastic limit.

i.e., Young's modulus (Y) $= \dfrac{\textbf{longitudinal stress}}{\textbf{linear strain}} = \dfrac{F/a}{l/L}$

∴ $\boxed{Y = \frac{FL}{al}}$ SI unit is Nm^{-2}

Where F- external force

L- Original length

l – Changing length

A – Area of cross section

(2) Rigidity modulus (n): it is defined as the ratio of tangential stress to shearing strain, with in elastic limit.

i.e., Rigidity modulus (n) = $\frac{\textbf{tangential stress}}{\textbf{shearing strain}} = \frac{F/a}{x/L} = \frac{F/a}{\theta}$

∴ $n = \frac{FL}{xa}$ or $\boxed{n = \frac{F}{a\theta}}$ SI unit is Nm^{-2}

Where F- external force

L- Original length

x – Displacement of height

A – Area of surface

(3) Bulk modulus (K): it is defined as the ratio of compressive stress to volume strain, with in elastic limit.

i.e., Bulk modulus (K) = $\frac{\textbf{compressive stress}}{\textbf{volume strain}} = \frac{F/a}{v/V}$

∴ $\boxed{K = \frac{FV}{va}}$ SI unit is Nm^{-2}

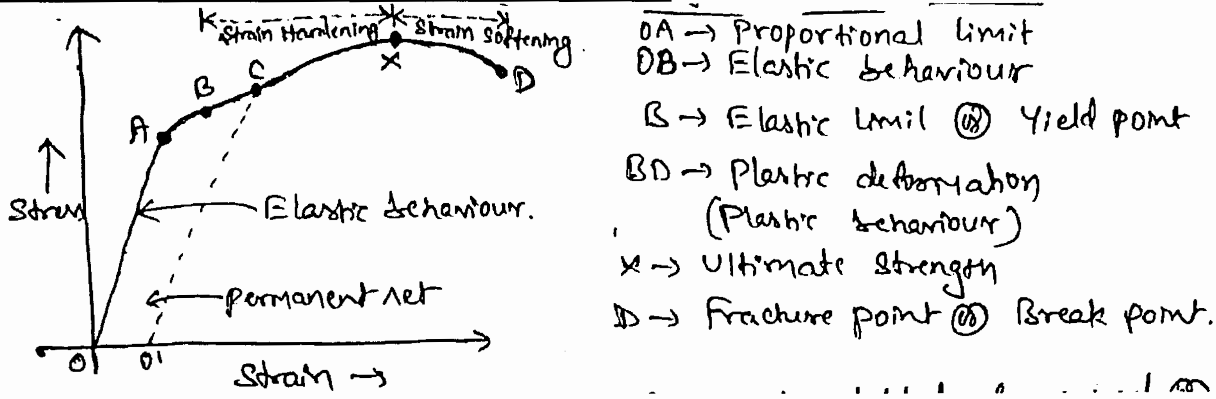
Where F- external force

V- Original volume

v – Changing volume

A – Area of surface

<u>**Stress - Strain curve (variation of strain with stress)**</u>



A graph of stress v/s strain is plotted for a rod or metallic wire.

When a wire is stretched by a load, as stress increase, strain also increases. In the graph region OA indicates the proportionality region i.e. as stress increases strain also increases and the body regains its original size or shape

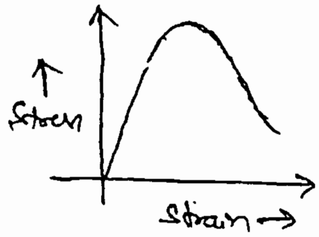
As stress is further increased, there is a large strain even for a small stress up to the point B. When the load is gradually removed between the points B to D, the wire returns to its original length and regains its dimension. This is known as a elastic limit or Yield point (B) OB region shows elastic behavior.

If the stress or load is increased beyond the point B the strain increases in large amount represented by the region BC. If the load is removed, the wire does not regain its original length but increases its length, even when stress is zero. This leads to permanent set represent by OO^1.

The wire experience large strain even for small stress beyond the point C. Hence stress attains maximum value at the point X, known as ultimate strength due to large plastic deformation. Beyond the point X, both stress and strain decreases lead to fracture of material. The point D is called fracture point.

The region B to X is known as strain hardening region, which indicates strengthening of material by plastic deformation.

The region X to D is known as strain softening region, which indicates weakening of the material by large plastic deformation or dislocations.



Strain hardening: is a process of making a metal harder by plastic deformation. Strain hardening is the effect of increasing yield point when a plastically deformed body is furthers subjected to stress. It is also known as cold working or work hardening.

In the graph, the curve OC indicates the variation of stress versus strain for a fresh specimen/metal. When the stress is applied beyond point B (elastic limit) then the body turns to plastic region and shows permanent set the presented by CC^1.

When the fresh specimen/metal is subjected to stress beyond the yield point, then stress and strain varies linearly up to the point P starting from the origin O^1. Hence yield point is increased due to large plastic deformation, which leads to hardening.

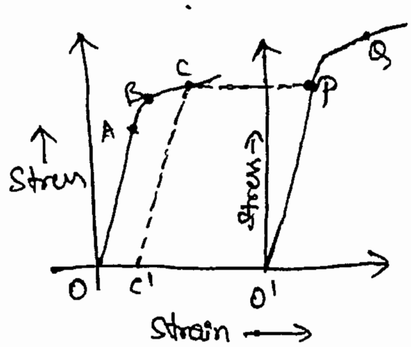
The yield point of a plastically deformed material can be increased by aligning the dislocations of atoms or molecules in line due to application of stress.

The dislocations or irregularity arises when is small group of atoms skip their positions and aligned in different lines. The stress applied in the plastic region, makes this dislocations move closer and hence atoms or molecules are compressed at certain regions and elongated in the neighboring regions. This close arrangement leads to strength. By adding impunity or subjected to annealing, these dislocated atoms can be brought into same line and hence metal becomes stronger. Hence strain hardening is achieved.

Strain softening: is a process of making the metal weaker by applying the stress beyond ultimate strength in plastic region.

When that applied stress increases to high value then, this may results in large plastic dislocations of molecules and material becomes weak and this leads to break or fracture.

In strain softening process, stress strain curve shows a negative slope as soon as the material crosses the yield point. Dislocations can easily move and break the material. Example: soil or concrete shows a negative slope.



Fatigue/Fracture- Failure of Elastic behavior:

The loss of strength of the material due to repeated strains on the material is called elastic fatigue.

When machines, parts of automobile, pumps, aircraft, turbines, compressors are subjected to continuous or repeated loading while in use, the cracks are formed due to repeated stress and this leads to fracture.

The fatigue is mainly due to repeated stress and also due to other variables such as overload, temperature, corrosion, structural defects, etc.

There are two types of fractures, namely

(a) **Brittle fracture:** A substance which breaks or undergoes fracture as soon as it crosses elastic limit is called brittle substance. Brittle fracture is mainly due to sudden propagation of cracks in a material.

(b) **Ductile fracture:** In a bulk substance from which sheets, wires can be drawn without affecting the elasticity. Ductile fracture is mainly arises due to slow propagation of cracks and accumulation of all the micro cracks to form big central crack.

Poisson's ratio (σ): Poisson's ratio is widely used in elasticity and it is a pure number and has no unit.

Poisson's ratio is defined as the ratio of lateral strain to the longitudinal strain.

$$\text{i. e, Poisson's ratio, } (\sigma) = \frac{\text{lateral strain}}{\text{longitudinal strain}} \quad \text{------------------------ (1)}$$

Lateral strain is defined as the ratio of change in diameter (d) to the original diameter (D).

$$\text{i. e, Lateral strain} = \frac{\text{change in diameter}}{\text{original diameter}} = \frac{d}{D} \quad \text{------------------------ (2)}$$

Longitudinal strainis defined as the ratio of change in length (l) to the original length (L).

$$\text{i. e, Longitudinal strain} = \frac{\text{change in length}}{\text{original length}} = \frac{l}{L} \quad \text{------------------------ (3)}$$

put Equation (2) & (3) in (1)

$$\text{Equation (1)} \Rightarrow \sigma = \frac{d/D}{l/L}$$

$$\boxed{\sigma = \frac{Ld}{lD}} \quad \text{------------------------ (4)}$$

Longitudinal strain coefficient (α): longitudinal strain produced per unit stress is called longitudinal strain coefficient.

If T is the applied stress and l/L is the longitudinal strain,

$$\text{Then, longitudinal strain coefficient}(\alpha) = \frac{\text{longitudinal strain}}{\text{applied stress}}$$

$$\therefore \alpha = \frac{l/L}{T} \quad \Rightarrow \quad \alpha = \frac{l}{TL} \quad \text{------------------------ (1)}$$

Or Extension produced, $l = \alpha TL$ ------------------------ (2)

Lateral strain coefficient (β): lateral strain produced per unit stress is called lateral strain coefficient.

If T is the applied stress and d/D is the lateral strain,

$$\text{Then, lateral strain coefficient } (\beta) = \frac{\text{lateralstrain}}{\text{applied stress}}$$

$$\therefore \beta = \frac{d/D}{T} \quad \Rightarrow \quad \beta = \frac{d}{TD} \quad \text{------------------------ (3)}$$

Or Contraction produced, $d = \beta TD$ ------------------------ (4)

Expression for Poisson's ratio (σ)in termsof (α) & (β):

Consider the ratio $\frac{\beta}{\alpha}$

Substituting using Eqn (1) & (2), $\frac{\beta}{\alpha} = \frac{\frac{d}{TD}}{\frac{l}{TL}} = \frac{dLT}{lDT}$

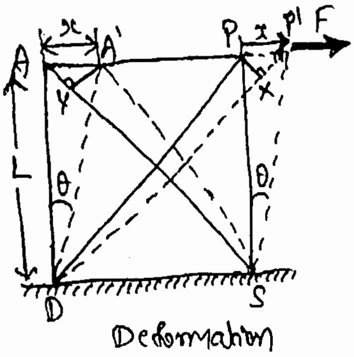
$\Rightarrow \quad \frac{\beta}{\alpha} = \frac{dL}{lD}$ ---------------------- (5)

But, **Poisson's ratio,** $\sigma = \frac{Ld}{lD}$ ---------------------- (6)

Comparing Equation (5) & (6), $\sigma = \frac{\beta}{\alpha}$ ---------------------- (7)

Using α& β, the expressions for Young's modulus (Y), Rigidity modulus (n) and Bulk modulus (k) can be obtained.

Relation between shearing strain, Elongation strain and compression strain (to show shear strain = longitudinal strain + compression strain):



Consider a cube of length L whose lower surface is fixed and a tangential force F is applied at the upper surface.

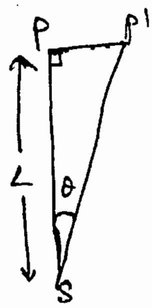
Let APSD be the one of the face of the cube having diagonals AS & DP. When deforming force F is applied on upper surface AP then point A slides to $A^1$and point P slides P^1 shown in the figure.

Let θ be the angle of shear and x is the displacement of the points. The diagonal AS contracts to A^1S and diagonal DP elongates to DP^1 by a distance x as shown in the figure.

Let a perpendicular PX drawn to diagonal DP^1 and A^1Y drawn to diagonal AS.

∴ DP = DX & A^1S = YS

Hence P^1X is the extension length along original length PD and represents Elongation strain = $\frac{\text{increase in length}}{\text{original length}} = \frac{P^1X}{PD}$ ---------------------- (1)



AY is the contraction in length along original length AS and represents Compression strain= $\frac{\text{decrease in length}}{\text{original length}} = \frac{AY}{AS}$ ---------------------- (2)

Sum of Equation (1) & (2) gives Shear Strain(θ).

Pythagoras theorem applied to triangles ADS or PSD

$\Rightarrow \mathbf{AS^2 = AD^2 + DS^2 = L^2 + L^2 = 2L^2}$

$\Rightarrow \mathbf{AS^2 = \sqrt{2}L = PD}$ ---------------------- (3)

From $\Delta^{le}PXP^1$, $\cos\angle PP^lX = \frac{P^1X}{PP^1}$

$\Rightarrow \mathbf{P^1X =} \ PP^l\cos\angle PP^lX$ ----------------------(4)

Since Δ^{le}APD is a isosceles triangle, $\angle\mathbf{APD = 45^0}$

If θ is small, then $\angle\mathbf{AP^1D} = \angle\mathbf{APD = 45^0}$

∴ Equation (4) becomes, $\mathbf{P^1X} = PP^l \cos(\mathbf{45^0}) = \frac{PP^1}{\sqrt{2}}$

$\Rightarrow \mathbf{P^1X} = \frac{x}{\sqrt{2}}$ ---------------------- (5)

From $\Delta^{le}PP^1S$, $\tan\theta = \frac{PP^1}{PS}$

For small angles, tanθ≈ **θ**

∴ $\theta = \frac{x}{L}$ ---------------------- (6)

put Equation (3) & (5) in (1),

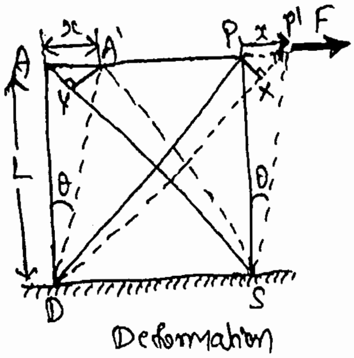
Eqn (1) ⇒ Elongation strain $= \frac{\frac{x}{\sqrt{2}}}{\sqrt{2}L} = \frac{x}{2L} = \left(\frac{x}{2}\right)\frac{1}{L} = \theta\left(\frac{1}{2}\right)$ (using Eqn-6)

∴ Elongation strain $= \frac{\theta}{2}$

Similarly we can show, Compression strain $= \frac{\theta}{2}$

∴Elongation strain + Compression strain $= \frac{\theta}{2} + \frac{\theta}{2} =$ **θ = Shear strain.**

Relation between Young's modulus (Y), Rigidity modulus (n) and Poisson's ratio(σ) [Y, n &σ]:



Consider a cube of length L whose lower surface is fixed and a tangential force F is applied at the upper surface.

Let APSD be the one of the face of the cube having diagonals AS & DP. When deforming force F is applied on upper surface AP then point A slides to A^1 and point P slides P^1 shown in the figure.

Let θ be the angle of shear and x be the displacement of the points. The diagonal AS contracts to A^{1}S and diagonal DP elongates to DP^1 by a distance x as shown in the figure.

Let a perpendicular PX drawn to diagonal DP^1 and A^{1}Y drawn to diagonal AS.

∴ DP = DX & A^{1}S = YS

Shear strain (θ) occurring along AP can be treated as equivalent longitudinal strain along the diagonal DP^1 and an equal lateral strain along diagonal A^{1}S that is perpendicular to DP.

If α is the longitudinal strain coefficient, then the extension producer for length DP due to tensile stress = DP α T ------------------------(1)

If β is the lateral strain coefficient, then the compression producer for length DP due to compressive stress = DP β T ------------------------ (2)

Where, T is the applied stress.

∴ Total extension along DP = DP T (α + β) (adding Eqns 1 & 2)

The diagonal DP changes to DP^1 due to stress,

Therefore P^{1}X is the total extension (see the figure)

∴ P^{1}X = DP T (α + β) ------------------------ (3)

But diagonal, DP = $\sqrt{2}$**L** and $\mathbf{P^1X} = \frac{x}{\sqrt{2}}$ (from figure)

∴ Eqn (3) ⇒ $\frac{x}{\sqrt{2}} = \sqrt{2}\mathbf{L\,T\,(\alpha + \beta)}$

$$\Rightarrow \frac{\mathbf{X}}{\mathbf{LT}} = \mathbf{2\,(\alpha + \beta)}$$

Inverting, $\frac{1}{2\,(\alpha+\beta)} = \frac{LT}{X} = \frac{T}{\frac{X}{L}} = \frac{T}{\theta} =$ **n (rigidity modulus)**

Where $\frac{X}{L} =$ **shear strain (θ)**

∴ $n = \frac{1}{2\,(\alpha+\beta)}$ ------------------------ (4)

Rearranging eqn(4), $n = \frac{1}{2\,\alpha(1+\frac{\beta}{\alpha})} = \frac{1}{2\,\alpha(1+\sigma)}$ (Since $\frac{\beta}{\alpha} = \sigma$, **Poisson'sratio**)

∴ $n = \frac{\frac{1}{\alpha}}{2\,(1+\sigma)}$ ------------------------ (5)

Young's modulus, $Y = \frac{\text{sress}}{\text{longitudinal strain}} = \frac{1}{\frac{\text{longitudinal strain}}{\text{stress}}} = \frac{1}{\frac{\text{strain along DP}}{\text{unit stress}}} = \frac{1}{\alpha}$

$\therefore \quad Y = \frac{1}{\alpha}$ ------------------------ (6)

Put Eqn (5) in (6),

Eqn (6) ⇒ $n = \frac{Y}{2(1+\sigma)}$ ------------------------ (7)

Or $\boxed{Y = 2n(1+\sigma)}$ ------------------------ (8)

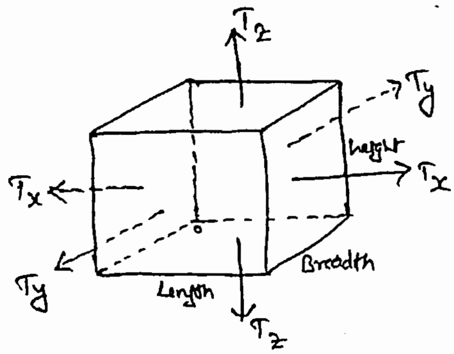
Equation (8) is the relation between $\mathbf{Y, n}$ & $\boldsymbol{\sigma}$

Relation between Young's modulus (Y), Bulk modulus (K) and Poisson's ratio(σ) [Y, K &σ]:

consider cube of length of unit length, breadth and height. Let T_x, T_y & T_z be the applied stress acting outwards along x, y & z directions respectively.

Let α be the elongation per unit length per unit stress in the direction of force T_x and β be the contraction per unit length per unit stress in the direction perpendicular to respective forces.

The applied stress T_x produces increase in length of αT_x in x-direction and there is a contraction in length of βT_y and βT_z along y and z directions respectively.



Volume deformation in cube

$\therefore$ Change in length of the cube along x-axis = $1+\alpha T_x - \beta T_y - \beta T_z$
Change in breadth of the cube along y-axis = $1+\alpha T_y - \beta T_z - \beta T_x$ --------(1)
Change in height of the cube along z-axis = $1+\alpha T_z - \beta T_x - \beta T_y$

Where 1 indicates original volume (unit volume)

$\therefore$ New volume of the cube = change in length x change in breadth x change in height

Using equation (1),

New volume = $(1+\alpha T_x - \beta T_y - \beta T_z) + (1+\alpha T_y - \beta T_z - \beta T_x) + (1+\alpha T_z - \beta T_x - \beta T_y)$

If α&β are small, then their product and powers are still small. Therefore neglecting the terms $\alpha\beta, \alpha^2, \beta^2$ etc, the above equation can be written as

New volume = $1+\alpha T_x + \alpha T_y + \alpha T_z - 2\beta T_x - 2\beta T_y - 2\beta T_z$

⇒ New volume = $1+\alpha(T_x + T_y + T_z) - 2\beta(T_x + T_y + T_z)$ ------------------------(2)

$\therefore$Increase in volume = New volume - Original volume

Increase in volume = $[1+\alpha(T_x + T_y + T_z) - 2\beta(T_x + T_y + T_z)] - 1$

If the applied stress are equal then $T_x = T_y = T_z = T$

Then Increase in volume = $\alpha(3T) - 2\beta(3T)$

⇒ Increase in volume = $3T(\alpha - 2\beta)$ ------------------------ (3)

If P is the inward pressure applied instead of outward stress, then the

Decrease in volume = $3P(\alpha - 2\beta)$ ------------------------ (4)

$\therefore$ Volume strain = $\frac{\text{decrease in volume}}{\text{original volume}} = \frac{3P(\alpha-2\beta)}{1}$ ------------------------ (5)

But Bulk modulus, $K = \frac{\text{pressure}}{\text{volume strain}} = \frac{P}{3P(\alpha-2\beta)}$

$\therefore \quad K = \frac{1}{3(\alpha-2\beta)}$ ------------------------ (6)

Rearranging Eqn (6), $K = \frac{1}{3\alpha(1-2\frac{\beta}{\alpha})} = \frac{\frac{1}{\alpha}}{3(1-2\sigma)} = \frac{Y}{3(1-2\sigma)}$ (Since $\frac{\beta}{\alpha} = \sigma$ & $\frac{1}{\alpha} = Y$)

∴ $$K = \frac{Y}{3(1-2\sigma)}$$ ------------------------ (7)

Equation (7) is the relation between **K, Y &σ**

Relation between K, n & Y:

We know the relation between **Y, n &σ** is Y = 2n (1+ **σ**) ------------------------ (1)

Rewriting eqn (1) $\frac{Y}{n} = 2(1+\sigma)$

$\frac{Y}{n} = 2 + 2\sigma$ ------------------------ (2)

Relation between K, Y and **σ** is $K = \frac{Y}{3(1-2\sigma)}$ ------------------------(3)

Rewriting eqn (3), $\frac{Y}{3K} = 1 - 2\sigma$ ------------------------(4)

Eqn (3) + (4) ⇒ $\frac{Y}{n} + \frac{Y}{3K} = 2 + 2\sigma + 1 - 2\sigma = 3$

$$\Rightarrow Y\left[\frac{3K+n}{3Kn}\right] = 3$$

⇒ $$Y = \frac{9Kn}{3K+n}$$ ------------------------ (5)

Relation between K, n &σ:

We know Y = 2n (1+ **σ**)

⇒ Y =$2n + 2n\sigma$) ------------------------(1)

We know $Y = 3K(1-2\sigma)$

⇒ $Y = 3K - 6K\sigma$ ------------------------ (2)

Equating eqn (1) & (2),

$2n + 2n\sigma = 3K - 6K\sigma$

$2n\sigma + 6K\sigma = 3K - 2n$

Simplifying, we get

⇒ $$\sigma = \frac{3K-2n}{2n+6K}$$ ------------------------ (3)

Limiting Value of σ (Poisson's ratio):

We know the relation between **Y, n &σ** is Y = 2n (1+ **σ**)

Relation between K, n &**σ is** $Y = 3K(1-2\sigma)$

⇒ 2n (1+ **σ**) = $3K(1-2\sigma)$ ------------------------ (1)

If **σ** is given positive value then LHS of Eqn (1) is positive & RHS of Eqn (1) will be positive only if **σ** takes values less than $\frac{1}{2}$ or 0.5.

If **σ** takes more than 0.5 value then RHS of Eqn (1) becomes negative

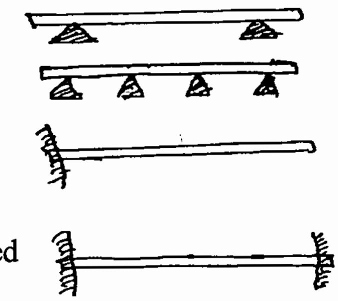
Therefore **σ** can take values between 0 and 0.5 to become a positive value.

If **σ** takes negative value then RHS will be positive only if **σ** takes value less than -1. If **σ**> -1 then LHS becomes negative. Therefore **σ** lies between -1 & 0.5.

If **σ** is negative value then we get lateral elongation instead of compression of a body, which is practically not possible. Therefore **σ** cannot take negative value.

∴ **σ** Can take values between 0 and 0.5

Beams: is a homogeneous body having uniform cross section, whose length is large compared to other dimensions like breadth, thickness etc, Types of beams:

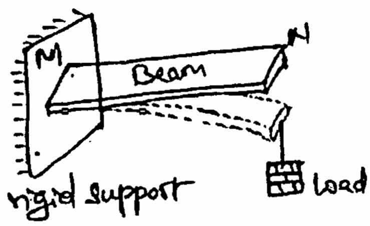


1. **Simple beam:** is a bar resting on two supports at its ends. It is the most commonly used to beam.
2. **Continuous beam:** is a bar resting on more than two supports.
3. **Cantilever beam:** is a bar or beam whose one end is fixed and other end is free.
4. **Fixed beam:** A beam which is fixed at its both ends is called fixed beam.

Engineering applications of Beams:

(a) Construction of bridges and plat forms.
(b) Elevators.
(c) Chassis/ frame as truck beds.
(d) Fabrication of trolley wheels.
(e) I – shaped gridersin buildings and bridges.

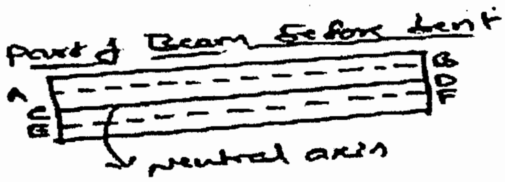
Bending of Beams: when a long beam is fixed at one end and loaded at the other end then bending moment is observed which is very large compared to shear stress.

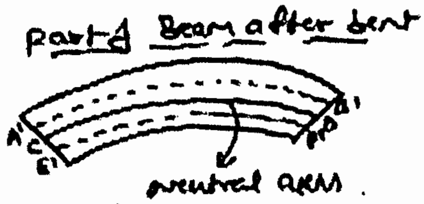


A single cantilever beam is a beam whose one end is fixed to rigid support and other end is loaded.

A uniform beam MN can be though made up of number of parallel layers and each layer intern considered to be made up of infinitesimally thin straight parallel longitudinal filaments arranged close to each other.

Neutral surface and neutral axis:

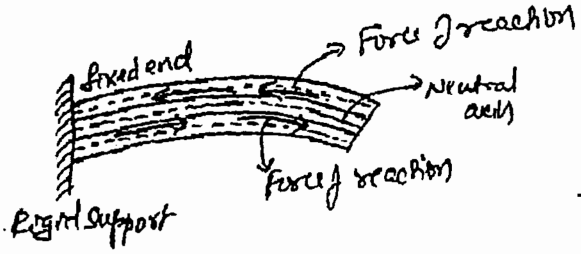




Let us consider the part of beam subjected to bent by applying load. In diagram layer CD does not bend under strain even when load is applied. This layer is considered to be neutral surface. Above the layer CD, the layer AB elongated to A^1B^1 and below the layer CD, the layer EF contracted to E^1F^1 as shown in figure. Therefore neutral surface is taken as a reference layer.

Neutral surface: It is a layer of uniform beam which is not undergoes any change in its dimension when the beam is subjected to bending within its elastic limit.

Neutral axis: It is a longitudinal line along neutral surface in a plane of bending of beam.

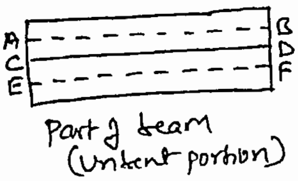


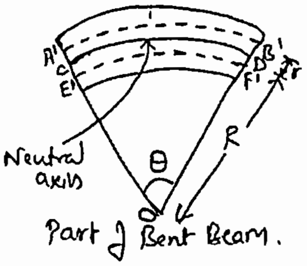
The layer above neutral surface elongates and experience inward pull (force of reaction) towards rigid support and the layers below neutral axis contracts or compress and shows outward push (force of reaction) away from the rigid support. These couple of forces causing restoring moment of beams and balances the load under equilibrium condition.

Bending moment of beam:

It is the moment of the applied couple subjected to which the beam under goes bending longitudinally.

It is also known as moment of force. When the beam is in equilibrium the bending moment and the restoring moment are equal.





Consider a long uniform beam whose one end is fixed and other end this load is attached. Due to the applied load the successive layers like AB, CD & EF are strained. The layer CD is neutral surface which does not change its length. The layer AB which is above the neutral surface will be elongated to A^1B^1 and layer EF below the neutral surface contracted to E^1F^1.

The bent beam is imagining forming a part of concentric circles of different radii of different layers. Let R be the radius of circle to which the neutral surface CD forms a part. Therefore Length of arc CD = Rθ (AB = EF = Rθ for unbent portion)

Where θ is the common angle subtended by layers at a common center O of the circle. If r is the distance of layer A^1B^1 from the neutral surface CD, then length of the arc A^1B^1 will be (R + r) θ

∴ Change in length = A^1B^1 – AB

$= (R + r)\,\theta - r\theta$

Change in length = rθ ------------------------ (1)

$\therefore$ **Longitudinal strain** $= \dfrac{\text{change in length}}{\text{original length}} = \dfrac{r\,\theta}{R\,\theta} = \dfrac{r}{R}$ ------------------------ (2)

We know, Young's modulus (Y) $= \dfrac{\text{longitudinal stress}}{\text{linear strain}} = \dfrac{F/a}{r/R}$

$\therefore \quad Y = \dfrac{FR}{ra}$ SI unit is Nm^{-2}

Where a- area of layer AB

F- Force acing on the beam

$\therefore \quad$ **Force,** $F = \dfrac{Yar}{R}$ ------------------------ (3)

Moment of force about neutral axis = Force x distance from neutral axis

Moment of force = F x r ------------------------ (4)

Put Eqn(3) in (4),

Eqn (4) ⇒ Moment of force $= \dfrac{Yar}{R}.r = \dfrac{Yar^2}{R}$

Moment of force of layer CD= $\dfrac{Yar^2}{R}$ ------------------------(5)

∴ Moment of force acting on the entire beam $= \sum \dfrac{Yar^2}{R}$

Moment of force $= \dfrac{Y}{R}\sum ar^2$ ------------------------ (6)

If $\sum mr^2$ is the moment of inertia of the beam, then we can consider $\sum ar^2 = I_g$ as geometric moment of inertia.

∴ Eqn (6) ⇒Moment of force $= \dfrac{Y}{R} I_g$

Or $\boxed{\textbf{Bending moment} = \dfrac{Y}{R} I_g}$ ------------------------ (7)

Note (1): Expression for bending moment of a rectangular beam:

For rectangular beam, $\mathbf{I_g} = \frac{bd^3}{12}$

$\therefore$ Eqn (7) $\Rightarrow$ **Moment of force** $= \frac{Y}{R}\frac{bd^3}{12}$

Where b – breadth of beam

Y – Young's modulus

d – Thickness of beam

Note (2): Expression for bending moment of a circular beam:

For circular beam, $\mathbf{I_g} = \frac{\pi r^2}{4}$

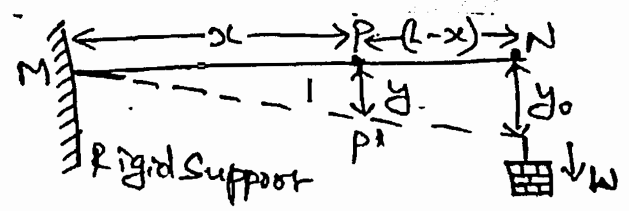
$\therefore$ Eqn (7) $\Rightarrow$ $\boxed{\textbf{Moment of force} = \frac{Y}{R}\frac{\pi r^2}{4}}$

Where r – radius of circular beam

Y – Young's modulus

Single cantilever: single cantilever is a beam (Rectangular shape) fixed at one end and loaded at the other end.

Expression for Young's modulus/Depression of single cantilever:



Considered a single cantilever of length L whose one end (M) is fixed to rigid support other end (N) is loaded (W)

Let p be a point at a distance x from the fixed end and (L-x)be the distance of the same point from the loaded end N. due to the load W the p bends to p^I. Let y^I be the depression at the point P

Therefore moment of force or bending moment = Force x perpendicular distance

$\therefore$ Bending moment = W (L-x) ------------------------(1)

But bending moment of beam= $\frac{Y}{R}\mathbf{I_g}$ ------------------------(2)

From Eqns (1) & (2), $\frac{Y}{R}\mathbf{I_g} = \mathbf{W(L-x)}$

$\Rightarrow \frac{1}{R} = \frac{W(L-x)}{YI_g}$ ------------------------ (3)

Where R is the radius of the circle for which the bent beam is the part

If y is the depression or the point P the distance x then we can show that rate of change of slope in second order is

$\frac{1}{R} = \frac{d^2y}{dx^2}$ ------------------------ (4)

Equating Eqn (3) & (4), $\frac{d^2y}{dx^2} = \frac{W(L-x)}{YI_g}$

$\Rightarrow \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{W(L-x)}{YI_g}$

$\Rightarrow \quad d\left(\frac{dy}{dx}\right) = \frac{W(L-x)}{YI_g}dx$

$\Rightarrow \quad d\left(\frac{dy}{dx}\right) = \frac{W(Ldx-xdx)}{YI_g}$ ------------------------ (5)

Integrating on both side of Eqn(5)

Eqn (5) $\Rightarrow \quad \frac{dy}{dx} = \frac{w}{YI_g}\left[Lx - \frac{x^2}{2}\right] + C_1$ ------------------------ (6)

In Eqn(6), C_1 is the integration constant. If y is the depression and x is the distance then at point x = 0, y = 0 $\therefore C_1 = 0$

Eqn (6) $\Rightarrow \frac{dy}{dx} = \frac{w}{YI_g}\left[Lx - \frac{x^2}{2}\right]$

$$\Rightarrow \quad dy = \frac{w}{YI_g}\left[Lx - \frac{x^2}{2}\right]dx$$

$$\Rightarrow \quad dy = \frac{w}{YI_g}\left[Lxdx - \frac{x^2}{2}dx\right] \quad \text{------------------------ (7)}$$

Integrating on both side of Eqn(7)

We get, $$y = \frac{w}{YI_g}\left[\frac{Lx^2}{2} - \frac{x^3}{6}\right] + C_2 \quad \text{------------------------ (8)}$$

Where C_2 is the integration constant and $C_2 = 0$ because at x = 0, y = 0

$\therefore$ Eqn (8) $\Rightarrow$ $$y = \frac{w}{YI_g}\left[\frac{Lx^2}{2} - \frac{x^3}{6}\right] \quad \text{------------------------ (9)}$$

For complete length of cantilever beam, x varies from x = 0 to x = L

$\therefore$ Eqn (9) $\Rightarrow$ $$y = \frac{w}{YI_g}\left[\frac{L^3}{2} - \frac{L^3}{6}\right] = \frac{wL^3}{YI_g}\left[\frac{1}{2} - \frac{1}{6}\right] = \frac{wL^3}{YI_g}\left[\frac{1}{3}\right]$$

$\therefore$ Depression, $$\boxed{y = \frac{wL^3}{3YI_g}} \quad \text{------------------------ (10)}$$

Young's modulus, $$\boxed{Y = \frac{wL^3}{3yI_g}} \quad \text{------------------------ (11)}$$

For rectangular beam of breadth (b), thickness (d), the Geometric moment of inertia, $I_g = \frac{bd^3}{12}$ and weight, W = mg

$\therefore$ Eqn (11) $\Rightarrow$ $$\boxed{Y = \frac{4mgL^3}{bdy}} \quad \text{------------------------ (12)}$$

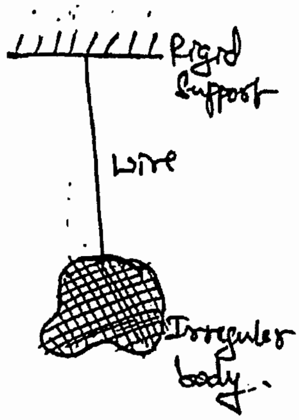
Torsional pendulum: It is a set up in which an elastic material in the form of wire executes to and fro motion along its length due to external couple, when one end of the body is fixed and other end attached to by a regular or irregular shaped rigid body.

The time Period of torsional pendulum for a regular rigid body suspended by a wire is calculated by

$$\boxed{T = 2\pi\sqrt{\frac{I}{C}}}$$

Where I - moment of inertia

C - Couple per unit twist



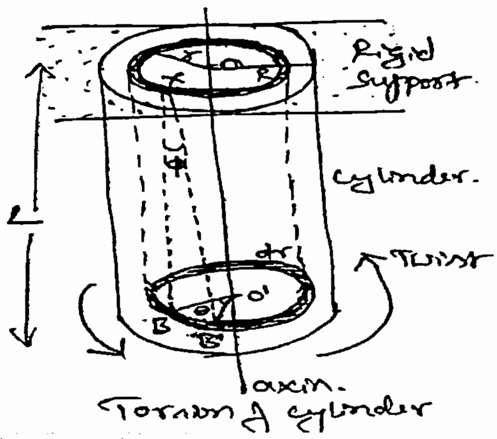
Applications of torsional pendulum:

(1) Rigidity modulus (n) of a material can be calculated using torsional pendulum

(2) The moment of inertia (I) of an irregular body can be calculated using torsional pendulum. Where as moment of inertia of a regular body can be calculated using mathematics.

Torsion of cylinder: A cylinder (long body) is set to be under tension, when it is twisted around its length as an axis.

This cylinder due to application of external couple restores its position due to elasticity.



Expression for Torsion of cylindrical rod:

In figure $\angle BXB^1 = \phi =$ **shearing angle**

$\angle BO^1B^1 = \theta =$ **twisting angle**

Consider a long cylindrical rod of length L and radius R which is fixed at one end (upper surface) to rigid support and external couple is applied to the lower end.

The cylinder is considered to made up of number of hallow coaxial cylindrical layers each of thickness dr. Let r be the radius of one such hallow cylinder.

When a couple is applied to lower surface, the point B shift to point B^1 through an angle ϕ which is known as angle of shear with respect to the point x, at the upper surface $\angle BXB^1 = \phi$ The cylinder twisted about an axis OO^1 therefore with respect to the point O^1, $\angle BO^1B^1 = \theta$

For small angles of ϕ, $BB^1 = L\phi$ ------------------------ (1)

For small angles of θ, $BB^1 = r\theta$ ------------------------ (2)

From Eqn (1) & (2), $L\phi = r\theta$

$\Rightarrow \phi = \frac{r\theta}{L}$ ------------------------ (3)

As couple is applied, the angle ϕ increases from fixed end to other end.

T is a shearing stress applied to the lower end of cylinder given by

$T = \frac{F}{a}$ ------------------------ (4)

a – area of cross section of thickness dr and given by

$a = 2\pi r.dr$ ------------------------ (5)

Put Eqn (5) in (4),

Eqn (4) $\Rightarrow$ $T = \frac{F}{2\pi r.dr}$

$\therefore$applied force, $F = T\, 2\pi r.dr$ ------------------------(6)

But Rigidity modulus is given by $n = \frac{T}{\phi}$

$\Rightarrow$ $T = n\phi$ ------------------------ (7)

Put Eqn(3) in (7)

Eqn (7) $\Rightarrow T = n\frac{r\theta}{L}$ ------------------------ (8)

Put Eqn(8) in (6)

Eqn (6) $\Rightarrow$ $F = n\frac{r\theta}{L}2\pi r.dr$

$\Rightarrow F = \frac{n\theta}{L}2\pi r^2.dr$------------------------ (9)

Moment of force leads to couple,

$\therefore$ Couple = Force x perpendicular distance

$\Rightarrow$ $c = F \times r = \frac{n\theta}{L}2\pi r^3.dr$ ------------------------ (10)

To find twisting couple of entire rod, Eqn (10) is integrated between the limits r = 0 to r = R

$\therefore$ Eqn (10) $\Rightarrow$ $c = \int_r^R \frac{n\theta}{L}2\pi r^3.dr = \frac{n\theta}{L}2\pi\int_r^R r^3.dr$

$$\Rightarrow c = \int_r^R \frac{n\theta}{L} 2\pi r^3 . dr = \frac{n\theta}{L} 2\pi \int_r^R r^3 . dr = \frac{n\theta}{L} 2\pi \left[\frac{R^4}{4} - 0\right]$$

$$\Rightarrow \quad c = \frac{\pi n\theta}{2L} R^4$$

$\therefore$Couple per unit twist, $C = \frac{c}{\theta} \Rightarrow \boxed{C = \frac{\pi n R^4}{2L}}$ ------------------------ (11)

Eqn (11) is the expression for Couple per unit twist.

QUESTION BANK
MODULE 2 –ELASTIC PROPERTIES OF MATERIALS

1. Explain the terms stress and strain. Discuss its types.
2. Define and explain Young's modulus, Bulk modulus, Rigidity modulus and Poisson's ratio.
3. Explain a perfectly elastic and plastic body with an example .What is the importance of elasticity in engineering applications.
4. State and explain Hooke's law. Explain the nature of elasticity with the help of stress-strain diagram.
5. Write a note on strain hardening and strain softening.
6. Discuss the effect of temperature, annealing, & impurities on elasticity.
7. Explain briefly the fracture or fatigue in elasticity.
8. Derive the relation between $Y, \eta \& \sigma$.
9. Derive the relation between $K, Y \& \sigma$.
10. Derive the relation between $K, \eta \& \sigma$ and also $K, \eta \& Y$ and discuss the limiting values of σ.
11. Explain the term bending moment of a beam. Obtain the expression for bending moment of a rectangular cross section and of a circular cross section beams.
12. Explain the neutral surface and neutral axis of beams .Mention different types of beams.
13. Obtain the expression for the depression at the free end of a single cantilever and hence write an expression for Young's modulus.
14. Derive the expression for couple per unit twist of a solid cylinder.
15. What is torsional pendulum? Give the expression for period of oscillation for a torsional pendulum.

NUMERICALS ON ELASTICITY

1. Calculate the force required to produce an extension of 1mm in steel wire of length 2 meters and diameter 1 mm. (Young's modulus for steel Y= 2 x 10^{11} N/m^2)

   Data: x=1mm, L=2 m, d=1 mm, Y=2 x 10^{11} N/m^2, F=?

   Solution: Radius of the wire $R = d/2 = \frac{1\times10-3}{2} = 0.5\times10^{-3}\ m.$

   Young's modulus $Y = \frac{FL}{ax} = \frac{FL}{\pi R^2 x}$

   $$F = \frac{\pi R^2 xY}{L} = \frac{3.14\times(0.5\times10^{-3})^2\times10^{-3}\times2\times10^{11}}{2} = 78.54\ N$$

2. Calculate the extension produced in a wire of length 2 m and radius 0.013 x 10^{-2} m due to a force of 14.7 N applied along its length. Given, Young's modulus of the material of the wire, Y= 2.1 x 10^{11} N/m^2.

   Data: x=? , L=2 m, F=14.7 N, Y=2.1 x 10^{11} N/m^2, R=0.013x10^{-2}m

   Solution: $Y = \frac{FL}{ax} = \frac{FL}{\pi R^2 x}$

   $$x = \frac{FL}{\pi R^2 Y} = \frac{14.7\times2}{3.14\times(0.013\times10^{-2})^2\times2.1\times10^{11}} = 2.6\times10^{-3}\ m$$

3. A 0.60 mm thick gold wire of length 1.12 m elongates by 1 mm, when stretched by a force of 160 gm and twist 1 radian when equal and opposite torques of 10x10^{-5}N are applied at its end. Find the value of Poisson's ratio. Given η=2.34x10^9N/m^2

   Data: L=1.12m, x=1mm, Θ= 1 radian, η=2.34x10^9N/m^2, τ=10x10^{-5}N, d=0.60 mm, m=160x10^{-3}kg, σ=?

   Solution: $Y = \frac{FL}{ax} = \frac{mgL}{ax} = \frac{mgL}{\pi R^2 x}$ (where R=d/2=0.60mm/2=0.30 mm)

   $$= \frac{160\times10^{-3}\times9.8\times1.12}{3.14\times(0.30\times10^{-3})^2\times1\times10^{-3}} = 6.21\times10^9\ \text{N/m}^2$$

   WKT, $Y = 2\eta(1+\sigma)$

   Since $\eta = \frac{Y}{2(1+\sigma)}$

   Therefore, $\sigma = \frac{Y}{2\eta} - 1 = \frac{6.21\times10^9}{2\times2.34\times10^9} - 1 = 0.326$

4. In stretching experiment, the extension produced in a wire for a load of 1.5 kg is 0.2x10^{-2}m. The length of wire is 2 m and its radius is 0.013x10^{-2} m. Find Young's modulus of the material of the wire.

Data: $x=0.2\times10^{-2}$ m, L=2 m, m=1.5 kg, Y=? $R=0.013\times10^{-2}$m

Solution: $Y = \frac{FL}{ax} = \frac{mgL}{\pi R^2 x} = \frac{1.5\times9.8\times2}{3.14\times(0.013\times10^{-2})^2\times0.2\times10^{-2}} = \mathbf{2.7\times10^{11}\ N/m^2}$

5. Calculate the torque required to twist a wire of length 1.5 m, radius 0.0425×10^{-2}m, through an angle $(\pi/45)$ radian, if the value of rigidity modulus of its material is 8.3×10^{10}N/m^2.

Data: L=1.5 m, $R=0.0425\times10^{-2}$m, $\Theta=(\pi/45)$ radian, $\eta=8.3\times10^{10}$N/m^2, τ=?

Solution: couple per unit twist $C = \frac{\pi\eta R^4}{2L}$

$= \frac{3.14\times8.3\times10^{10}\times(0.0425\times10^{-2})^4}{2\times1.5} = \mathbf{2.8357\times10^{-3}}$

Torque $\tau = C\,\Theta$

$= \mathbf{2.8357\times10^{-3}} \times \frac{3.14}{45} = \mathbf{1.98\times10^{-4}Nm}$

6. A wire of length 2 m and radius 0.2×10^{-2} m is fixed to the center of a wheel. A torque of magnitude 0.0395 Nm is applied to twist the wire .Find the angle of twist. Given the rigidity modulus of the wire $\eta=8.3\times10^{10}$N/m^2.

Data: L=2 m, $R=0.2\times10^{-2}$m, τ=0.0395 Nm, Θ=?, $\eta=8.3\times10^{10}$N/m^2

Solution: Torque $\tau = C\,\Theta$

Where, $C = \frac{\pi\eta R^4}{2L}$

Hence, $\tau = \frac{\pi\eta R^4}{2L}\Theta$

$\Theta = \frac{2\tau L}{\pi\eta R^4}$

$= \frac{2\times0.0395\times2}{3.14\times8.3\times10^{10}\times(0.2\times10^{-2})^4} = \mathbf{0.0378\,\mathit{radian}}$

7. A rectangular bar with 2 cm breadth 1 cm depth and 100 cm length is supported at its ends and a load of 2 kg is applied at its midpoint. Calculate the depression if the Young's modulus of the material of the bar is 1.25×10^{10}N/m^2

Data: L=100 cm, $b=2\times10^{-2}$m, d=1cm, m=2 kg, $Y=1.25\times10^{10}$N/m^2, y_0=?

Solution: $y_0 = \frac{4wL^3}{bd^3Y} = \frac{4mgL^3}{bd^3Y}$

$= \frac{4\times2\times9.8\times(100\times10^{-2})^3}{2\times10^{-2}\times(1\times10^{-2})^3\times1.25\times10^{10}} = \mathbf{0.3136\ m}$

EXERCISE PROBLEMS

8. Calculate the angular twist of a wire of length 0.3 m, and radius 0.2×10^{-3}m when a torque of 5×10^{-4} Nm is applied. Rigidity modulus of the material is 8.3×10^{10}N/m^2.

9. An increment in length by 1mm was observed in a gold wire of diameter 0.3 mm, when it was subjected to a longitudinal force of 2 N, and a twist of 0.1radian was observed in the same wire when its one end was subjected to a torque of 7.9×10^{-7} Nm, while other end was fixed. Calculate the value of Poisson's ratio for gold.

10. A brass bar of length 1m, breadth 0.01m and depth and 0.0145 m is clamped firmly in a horizontal position at one end. A weight of 1 kg is applied at the other end. What depression would be produced? Young's modulus of the material of the bar is 9.78×10^{10}N/m^2

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