Syllabus:

Module-1: Oscillations & Waves

Free Oscillations: Definition of SHM, derivation of equation for SHM. Mechanical simple harmonic oscillator (mass suspended to spring oscillator), Complex notation and phasor representation of simple harmonic motion. Equation of motion for free oscillations. Natural frequency of oscillations.

Damped oscillations: Theory of damped oscillations: over damping, critical & under damping, quality factor.

Forced oscillations : Theory of forced oscillations and resonance, Sharpness of resonance. One example for mechanical resonance.

Shock waves: Mach number, Properties of Shock waves, control volume. Laws of conservation of mass, energy and momentum Construction and working of Reddy shock tube, applications of shock waves.

OSCILLATIONS AND WAVES

Oscillations

Periodic motion: Motion that repeats itself its path after regular intervals of time is called periodic motion.

Examples:

1. Motion of planets around the sun
2. Motion of electrons around the nucleus, etc.

Oscillatory motion: The motion of a body is said to be oscillatory if it moves to and fro or back and forth about the mean position or equilibrium position after regular intervals of time. It is also known as vibratory motion.

Example:

1. To and fro motion of bob in a pendulum.
2. Mass attached to a spring.

Simple Harmonic Motion (SHM)

A body is said to be executing simple harmonic motion, if it moves to & fro about the mean position such that its acceleration is directly proportional to the displacement of the body from the mean position and direction of acceleration is opposite to the direction of displacement of the body.

Example: Motion of a mass attached to a spring.

Restoring force: The force which is developed in the body when it is displaced from the mean position is called restoring force.

Example: When a spring is compressed or elongated restoring force is developed (which always makes the spring to attain its original size or shape)

Restoring force = - (Applied force)

$$F = -F_{ext}$$

External force is proportional to displacement of a body

$$F_{ext} \propto x$$

$\Rightarrow \quad F \propto -x$

$\Rightarrow \quad F = -kx$

where k- indicate constant known as force constant or spring constant

$\Rightarrow \quad k = -\frac{F}{x}$

Force constant it is defined as restoring force per unit displacement. SI unit of force constant is Nm^{-1}

Differential equation for SHM and its solution

Consider a body of mass 'm' moving to & fro motion about mean position 'O' as shown in figure. Let 'F' be the restoring force and 'x' be the displacement of a body.

From Hooke's Law,

The restoring force, $F = -kx$ ---------------------- (1)

where k- spring constant.

From Newton's second law

Force, $F = ma$ ---------------------- (2)

but acceleration, $a = \frac{d^2x}{dt^2}$

∴ Eqn (2) ⟹ $F = m\frac{d^2x}{dt^2}$ ---------------------- (3)

From equations (1) and (3), $m\frac{d^2x}{dt^2} = -kx$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0} \quad \text{---------------------- (4)}$$

Let $\frac{k}{m} = \omega^2$

$$\therefore \text{Eqn (4)} \Rightarrow \boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \quad \text{---------------------- (5)}$$

Where ω- angular velocity or angular frequency

Equation (4) or (5) is the differential equation for SHM.

The solution for the above equation is given by

$$x = A\sin(\omega t + \phi) \quad \text{---------------------- (6)}$$

Eqn (6) is the expression for SHM.

Time Period(T): time taken to complete one oscillation. Time period is given by

$$T = \frac{2\pi}{\omega}$$

But $\omega = \sqrt{\frac{K}{m}}$

∴ above equation becomes $T = \frac{2\pi}{\sqrt{\frac{K}{m}}}$

$$\Rightarrow T = 2\pi\sqrt{\frac{m}{K}} \quad \text{---------------------- (7)}$$

equation 1 is the expression for time period o mass m and spring constant K.

Characteristics of SHM

Displacement(x): Distance travelled by a body from its mean position at any instant of time

displacement $x = A\sin(\omega t + \phi)$

Where, A – amplitude (the maximum displacement of the body from the mean position)

Velocity (v): Rate of change of displacement of a body is called velocity.

$$\therefore v = \frac{dx}{dt} = \frac{d}{dt}(A\sin(\omega t + \phi)$$

After simplifying, we get $\mathbf{v = \omega\sqrt{A^2 - x^2}}$ ---------------------- (1)

. Case (i): At Mean position, x = 0, $\therefore v_{max} = \omega A$
Case (ii): At Extreme position, x = A, $v_{min} = 0$

Acceleration (a): Rate of change of velocity of body is called acceleration.

$$\therefore a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{d^2}{dt^2}(A \sin(\omega t + \phi)$$

After simplifying, we get $\mathbf{a = -\omega^2 x}$ ------------------------- (2)

Case (i): At Mean position, x = 0, $\therefore a_{min} = 0$
Case (ii): At Extreme position, x = A, $a_{max} = -\omega^2 A$

Phase (ϕ): The state of vibration of a particle executing SHM is called phase.
The term $(\omega t + \phi)$ is called phase.

Frequency (f): Number of oscillations executed by a body in one second

$$\mathbf{f = \frac{1}{T}}$$

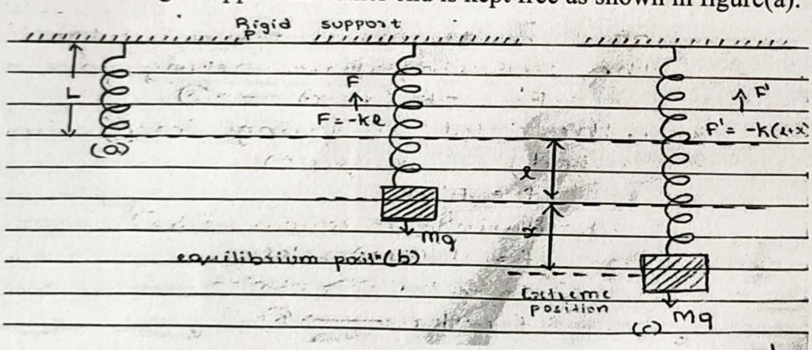
But angular frequency, $\boldsymbol{\omega = 2\pi f = \frac{2\pi}{T}}$

Since Time period, $T = \mathbf{2\pi\sqrt{\frac{m}{K}}}$

∴ Frequency, $\mathbf{f = \frac{1}{2\pi}\sqrt{\frac{K}{m}}}$ ------------------------- (3)

<u>Mechanical harmonic oscillator</u> - Mass suspended to Spring (vertical vibration):

Consider a spring of negligible mass and having length L let one of the end of the spring is attached to rigid support and other end is kept free as shown in figure(a).



Let a mass M is attached to its free end of the spring which will displace the spring by a distance l in download direction as shown in figure (b). Therefore from Hooke's law, the restoring force developed in a spring is given by

$$F = -k\,l$$ ------------------------- (1)

where k – spring or force constant.

The downward force mg acts opposite to the restoring force and balance the spring

∴ downward force, $F = -mg$ ------------------------- (2)

In equilibrium condition we can write

$$-mg = -k\,l$$

⇒ $mg = k\,l$

⇒ $mg = k\,l$

$$\frac{l}{g} = \frac{m}{k}$$ ------------------------- (3)

let the mass M is pulled downward through a distance x below the mean position to make vertical vibration. Now the restoring force acting on the mass is

$$F^{l} = - k (l + x) \quad \text{------------------ (4)}$$

Therefore, the total restoring force acting on the body of mass M

$$f = F^{l} - F - \quad \text{------------------ (5)}$$

Substituting equations (1) and (4) in eqn (5),
We get, $f = - k (l + x) + kl$

$$f = - kx \quad \text{------------------ (6)}$$

from Newton's second law, $f = ma$ ------------------ (7)
From Eqn (6) & (7), $ma = - kx$

$$\Rightarrow \quad a = \left(-\frac{k}{m}\right) x$$

Let $\frac{k}{m} = \omega^2$

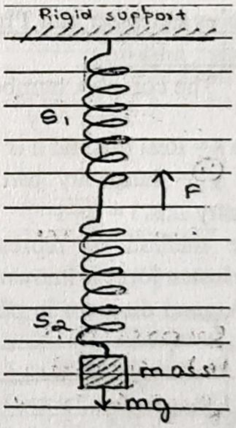
∴ Acceleration, $\boxed{a = -\omega^2 x}$ ------------------ (8)

Equestion (8) is this expression for acceleration of a body executing SHM.

Expression for spring force:
(Motion of a body suspended by two springs)

1. **In series combination** (expression for force constant in series combination):

Consider two springs S_1 and S_2 having Spring Constant K_1 and K_2 respectively connected into end in series combination. One end of the spring is attached to the rigid support and at the other free end the mass M is attached as shown in the figure. Since both the springs are connected in series combination therefore restoring force is same in both the springs which acts opposite to the applied force (mg)



If x_1 and x_2 be the displacement of the springs S_1 and S_2 respectively then restoring force acting on the spring S_1 is $F = - k_1 x_1$

$$\Rightarrow \quad x_1 = -\frac{F}{K_1} \quad \text{------------------ (1)}$$

The restoring force acting on the spring S_2 is $F = - k_2 x_2$

$$\Rightarrow \quad x_2 = -\frac{F}{K_2} \quad \text{------------------ (2)}$$

The total displacement of the series combination is $x = x_1 + x_2$

From Eqn (1) & (2), $x = -\frac{F}{K_1} - \frac{F}{K_2}$

$$x = -F\left(\frac{1}{K_1} + \frac{1}{K_2}\right)$$

$$\Rightarrow \quad x = -F\left(\frac{K_1+K_2}{K_1K_2}\right)$$

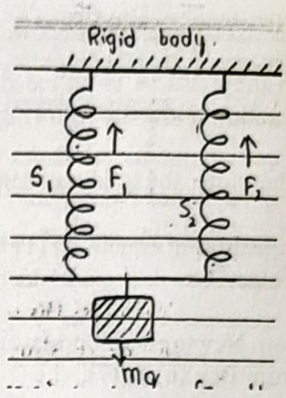
∴ Net restoring force is $F = -\left(\frac{K_1K_2}{K_1+K_2}\right)x$

Let $F = -K_S x$ ------------------ (3)

Where $\boxed{K_S = \frac{K_1K_2}{K_1+K_2}}$ ------------------ (4)

Equation (4) is expression for force constant in series combination

2. **In Parallel combination** (expression for force constant in parallel combination):



Consider two springs S_1 and S_2 having Spring Constant K_1 and K_2 respectively, attached to the same rigid support at different points. Let a body of mass M is attached to the free end of the springs, which makes parallel combination of two springs.

Since both the springs are attached in parallel combination, therefore displacement x remains the same for the applied load (mg).

The restoring force in the spring S_1 is

$$F_1 = - K_1 x \quad \text{------------------------ (1)}$$

The restoring force in the spring S_2 is

$$F_2 = - K_2 x \quad \text{------------------------ (2)}$$

Where, K_1 and K_2 are Spring Constant of springs S_1 and S_2 respectively.
The total force acting in parallel combination is

$$F = F_1 + F_2 \quad \text{------------------------ (3)}$$

Put eqn (1) & (2) in Eqn (3)

$\therefore$ Eqn (3) $\Rightarrow$ $F = - K_1 x \ - K_2 x$

$F = - x (K_1 + K_2)$

$\Rightarrow$ $F = - (K_1 + K_2)\, x$ ------------------------ (4)

Let $\mathbf{K_1 + K_2 = K_P}$ ------------------------ (5)

$\therefore$ Eqn (4) $\Rightarrow$ $F = - K_P\, x$ ------------------------ (6)

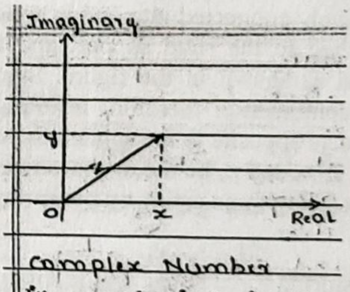
Where, $\boxed{K_P = K_1 + K_2}$ is expression for effective spring constant or force constant in parallel combination.

Complex notation and Phasor representation:

Complex notation:

The complex number Z is given by

$$Z = x + i\,y \quad \text{------------------------ (1)}$$

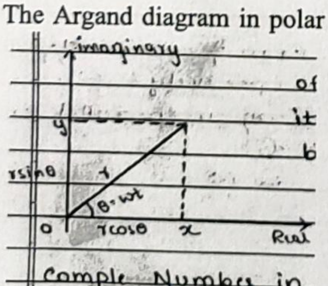


Where x = Real part and it is the component of z along real axis.

y = Imaginary part and it is the component of z along imaginary axis. $i = \sqrt{-1}$

The method of representation of complex number in a Cartesian form is known as **Argand diagram**.

The Argand diagram in polar form for complex number Z is as shown.



In polar form, the magnitude of Z is represented by r and the its position varies with time by angle **θ**.

$\therefore$ Eqn (1) $\Rightarrow$ $Z = r\cos\theta + i\, r\sin\theta$ ------------------------ (2)

Where,

$r\cos\theta$ = component of r along real axis

$i\sin\theta$ = component of r along imaginary axis.

From Euler's theorem,

$\mathbf{e^{i\theta} = cos\theta + isin\theta}$ -------------------------- (3)

$\therefore$ Eqn (2) $\Longrightarrow$ Z = r (cosθ + i sinθ) = $\mathbf{r\, e^{i\theta}}$

$\therefore$ $\mathbf{Z = r\, e^{i\theta}}$ -------------------------- (4)

If ω is the angular frequency, then the angular displacement $\theta = \omega t$.

$\therefore$ Eqn (4) $\Longrightarrow$ $\mathbf{Z = r\, e^{i\omega t}}$ ------------------------ (5)

At t = 0 second, is Z has some phase angle, ϕ

Then Eqn (5) $\Longrightarrow$ $\mathbf{Z = r\, e^{i(\omega t+\phi)}}$ ------------------------ (6)

Equation (6) is the expression for complex notation for a particle executing SHM.

If $\theta = 90^0$

Eqn (3) $\Longrightarrow$ $\mathbf{e^{i90} = cos90 + isin90}$

$\Longrightarrow$ $\mathbf{e^{i90} = i}$ ------------------------ (7)

Multiplying $\mathbf{e^{i\theta}}$ on both side of equation (7),

$\mathbf{e^{i90}e^{i\theta} = i\, e^{i\theta}}$

$\Longrightarrow$ $\mathbf{e^{i(90+\theta)} = i\, e^{i\theta}}$

$\Longrightarrow$ $\mathbf{i\, e^{i\theta} = e^{i(90+\theta)}}$

Which means multiplying i results in rotation of $\mathbf{90^0}$ in anticlockwise rotation.

$\therefore$ $\mathbf{\theta \to \theta + 90^0}$ is the condition-1

The velocity is the rate of change of displacement.

$\therefore \mathbf{v = \frac{dZ}{dt} = \frac{d}{dt}(r\, e^{i\omega t}) = r\, i\omega\, e^{i\omega t} = ir\omega\, e^{i\omega t} = i\, v_0\, e^{i\omega t}}$

$\Longrightarrow$ $\mathbf{v = e^{i90}\, v_0\, e^{i\omega t}}$

$\Longrightarrow$ $\mathbf{v = v_0\, e^{i(\omega t+90)}}$ ------------------------ (8)

Where $\mathbf{v_0} = r\omega$

From Eqn (8), velocity v is ahead of displacement r by an angle of $\mathbf{90^0}$

The acceleration is the rate of change of velocity.

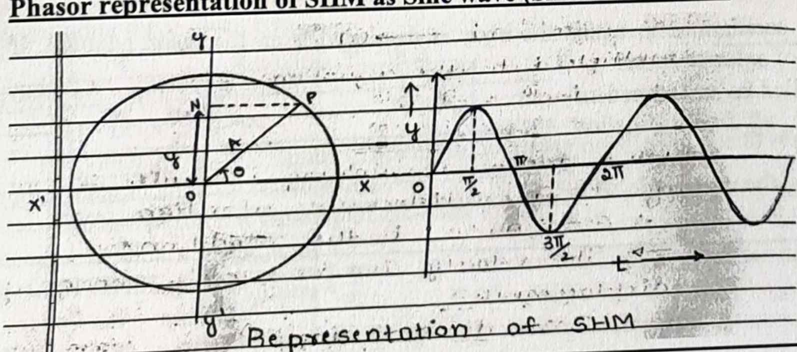
$\therefore \mathbf{a = \frac{dv}{dt} = \frac{d^2Z}{dt^2} = \frac{d^2}{dt^2}(r\, e^{i\omega t}) = r\, i^2\omega^2\, e^{i\omega t}}$

$\Longrightarrow$ $\mathbf{a = r\, i\, i\, \omega^2\, e^{i\omega t} = r\, e^{i90}e^{i90}\omega^2\, e^{i\omega t} = r\,\omega^2 e^{i(\omega t+180)}}$

$\Longrightarrow$ $\mathbf{a = a_0\, e^{i(\omega t+180)}}$ ------------------------ (9)

From Eqn (9), acceleration a is ahead of displacement r by an angle of $\mathbf{180^0}$

Phasor representation of SHM as Sine wave (Sinusoidal wave):



Consider a particle of mass 'm' moving on a circumference of a circle with constant angular velocity$\boldsymbol{\omega}$. Let the particle move from point x to point 'P' at time't' second by making an angle $\boldsymbol{\theta}$ in anticlockwise

direction. At point P, 'A' indicates the radius or the amplitude of the vector. The component of P along the y-axis is ON = y.
Where, y is the displacement of the particle at any instant of time.
As the particle moves along the circular path, we can observe to and fro motion along the y-axis.
∠Pox = $\boldsymbol{\theta}$, indicates Phase. The variation of y versus t is as shown in the graph.

Phasor representation is the complex representation of the magnitude and phase of a sinusoidal wave.

Hence complex notation for initial phase is $\mathbf{Z = r\, e^{i(\omega t+\phi)}}$

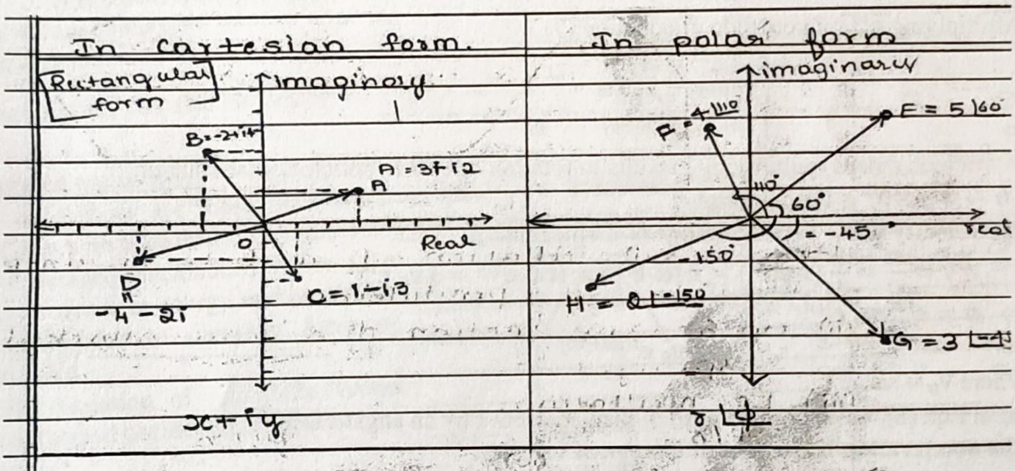
The arrow in the Argand diagram has a length r corresponding to the amplitude and $\boldsymbol{\theta = \omega t}$ gives the phase angle.

The rotating arrow Z is the phasor and it is represented by the equation $Z = r\,\angle\phi$

$$\Rightarrow \qquad r\,\angle\phi = r\cos\phi + i\,r\sin\phi$$

This is the Phasor representation of SHM.

Example for Phasor:

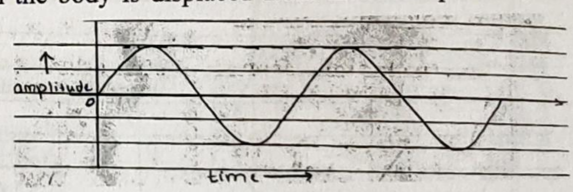


There are three different cases of vibration for SHM.

1. Free oscillation (Undamped oscillation)
2. Damped oscillation
3. Forced oscillation

1. **Free oscillation:**

Free oscillations are those oscillations in which the body is displaced from the mean position, it oscillate without influence of external force.



Free oscillation is also called as undamped or ideal oscillation. In nature, all free oscillation tends to damped oscillation.

In the free oscillation, the amplitude of a vibrating body remains constant for a long period as shown in the figure.

Example:

i) Oscillation of a bob in simple pendulum.

ii) Up and down motion of a mass attached in a spring.

iii) LC oscillation.

Natural frequency (ω), it is the frequency of a body which oscillate/vibrate without influence of external force.

Equation of motion for Free oscillation:

The general equation for SHM itself is the equation for free oscillation and it is given by

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

Where, x - displacement

K – Force constant

m - Mass

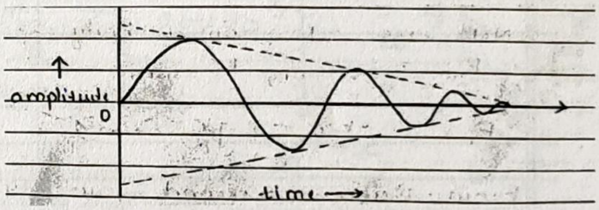
The above equation can also be written as $\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$ $\left(\because \frac{k}{m} = \omega^2\right)$

2. Damped oscillation:

Damped oscillations are those oscillations in which the amplitude of a vibrating body decreases exponentially with time due to combined effect of restoring force and resistive force.

In damped oscillations, there is a continues decrease in energy and amplitude of a particle due to damping force or frictional force.

The variation of amplitude versus time for a damped oscillation is as shown.



Example:

i) Oscillation of a bob in air medium in simple pendulum.

ii) Oscillation of a swing in air medium.

iii) Oscillation of mass attached to the spring whose lower part is immersed in liquid.

Theory of damped Oscillations:

Consider a body of mass m, executing oscillation. The vibrating body is constantly acted upon by restoring force given by,

$$\mathbf{F_{restoring} = -k\,x} \quad \text{-------------------------- (1)}$$

Where, k – force constant

x - displacement

The vibrating body decreases its amplitude during oscillation due to resistive force and it is proportional to velocity of the body.

Resistive force α velocity of the body in fluid

i.e, $F_{resitsive} \propto V$

$\Rightarrow$ $F_{resitsive} \propto \frac{dx}{dt}$

$$F_{resitsive} = -r\frac{dx}{dt} \quad \text{-------------------------- (2)}$$

Where, r – damping constant and it depends on nature of fluid.
Negative sign indicates Force, F is acts opposite to the displacement/ velocity.
∴ Net force acting on a body is

$$F_{Net} = F_{restoring} + F_{resitsive}$$

Substituting Eqn (1) & (2), we get, $F_{Net} = -k\,x - r\frac{dx}{dt}$ -------------------------- (3)

From Newton's second law, the net/resultant force is given by

$$F_{Net} = m\frac{d^2x}{dt^2} \quad \text{-------------------------- (4)}$$

Equating equation (4) and (3),

$$m\frac{d^2x}{dt^2} = -k\,x - r\frac{dx}{dt}$$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + k\,x = 0 \quad \text{-------------------------- (5)}$$

Divide by 'm' on each term of equation (5),

Eqn (5) ⟹ $\frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$ -------------------------- (6)

Let $\frac{r}{m} = 2b$ and $\frac{k}{m} = \omega^2$ (where **ω – natural frequency**)

Eqn (6) ⟹ $\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + \omega^2 x = 0$ -------------------------- (7)

Equation (7) is the expression for Equation of motion for Damped oscillation.
The solution of Eqn (7) is

$$x = A\,e^{\alpha t} \quad \text{-------------------------- (8)}$$

Where A and **α** are the arbitrary constants. To obtain A and **α**, Eqn (8) is differentiated twice.

Eqn (8) ⟹ $\frac{dx}{dt} = A\,\alpha\,e^{\alpha t}$

And $\frac{d^2x}{dt^2} = A\,\alpha^2\,e^{\alpha t}$ -------------------------- (9)

Substituting equation (9) in (7),

Eqn (7) ⟹ $A\,\alpha^2\,e^{\alpha t} + 2b\,A\,\alpha\,e^{\alpha t} + \omega^2 A\,e^{\alpha t} = 0$

⟹ $A\,e^{\alpha t}(\alpha^2 + 2b\alpha + \omega^2) = 0$ -------------------------- (10)

$\because A\,e^{\alpha t} \neq 0$ (otherwise, $x = 0$)

∴ $(\alpha^2 + 2b\alpha + \omega^2) = 0$ -------------------------- (11)

We know $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

∴ in Eqn (11) a = 1, b = 2b and c = ω^2

The standard solution for quadratic eqn (11) is $\alpha = \frac{-2b\pm\sqrt{4b^2-4\omega^2}}{2} = -b \pm \sqrt{b^2-\omega^2}$

∴ $\alpha = -b \pm \sqrt{b^2-\omega^2}$ -------------------------- (12)

∴Eqn (8) ⟹ $x = A\,e^{(-b\pm\sqrt{b^2-\omega^2})t}$ -------------------------- (13)

The general solution of equation (13) is

$$\boxed{x = C\,e^{(-b+\sqrt{b^2-\omega^2})t} + D\,e^{(-b-\sqrt{b^2-\omega^2})t}} \quad \text{-------------------------- (14)}$$

Eqn (14) is the general solution for Damped oscillation. Where C & D are constants and these values can be calculated initial and boundary conditions. They are given by

$$C = \frac{x_0}{2}\left[1 + \frac{b}{\sqrt{b^2 - \omega^2}}\right] \quad \text{and} \quad D = \frac{x_0}{2}\left[1 - \frac{b}{\sqrt{b^2 - \omega^2}}\right]$$

Where x_0 is the initial displacement at t = 0 sec.

In Eqn (13) and (14), the value of x depends on time't' and the term $\sqrt{b^2 - \omega^2}$

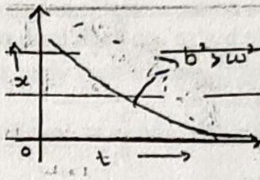
∴ We can get three cases.

Case (i): Over damping for $b^2 > \omega^2$

Case (ii): Critical damping for $b^2 = \omega^2$

Case (iii): Under damping for $b^2 < \omega^2$

Case (i): Over damping for $b^2 > \omega^2$:



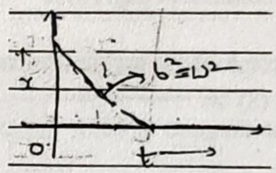
For Over damping, $b^2 > \omega^2$. ∴ $\sqrt{b^2 - \omega^2} = +ve$

⟹ $\sqrt{b^2 - \omega^2}$ is less than b.

∴ In Eqn (14), both the first term and second term are negative. Therefore displacement of a particle decreases exponential with time from maximum position after some time, as shown in the graph.

Example: Bob of a pendulum oscillating in castor oil.

Case (ii): Critical damping for $b^2 = \omega^2$:



For Critical damping, $b^2 = \omega^2$. If $b^2 = \omega^2$ then $\sqrt{b^2 - \omega^2} = \infty$

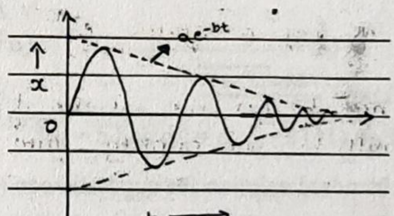
After simplifying we get, $x = e^{-bt}[P + Qt]$

Where P = C+D and Q = C-D

∴ In the above equation, the displacement x, of a particle decreases suddenly/ shortly with time as shown in the graph.

Example: Bob of a pendulum oscillating in water.

Case (iii): Under damping for $b^2 < \omega^2$:



For Under damping, $b^2 < \omega^2$. ∴ $\sqrt{b^2 - \omega^2} = -ve$

After simplifying we get, $x = a\,e^{-bt}[\sin(nt + \phi t)]$

Where $a = \frac{x_0 \alpha}{n}$ indicates amplitude and $n = \sqrt{\omega^2 - b^2}$

∴ In the above equation, the term $a\,e^{-bt}$ indicates amplitude of a particle and it decrease gradually as time increases and the variation of x v/s t is sinusoidal as shown in the graph.

Example: Bob of a pendulum oscillating in air medium.

Quality Factor:

Quality factor describes the amount of damping takes place in a vibrating body. The damping constant cannot be determined directly and also it is difficult to find the value of 'b'. Therefore we need to find a factor which describes how much under damping takes place in a oscillating system.

∴ We use a Quality factor to explain amount of damping and is given by the expression

$$Q = \frac{\omega}{2b}$$

Where, ω – natural frequency

$\Rightarrow \quad b = \frac{\omega}{2Q}$

Since, $n = \sqrt{\omega^2 - b^2} = \sqrt{\omega^2 - \frac{\omega^2}{4Q^2}} = \omega\sqrt{1 - \frac{1}{4Q^2}}$

Quality factor is defined as the number of cycles required for the energy to decay or fall off by a factor $e^{2\pi}$.

Significance: Larger the number of cycles more will be the quality factor.

3. Forced oscillation:

Forced oscillations are those oscillations in which the bodies vibrate/oscillate with the influence of periodic external force to overcome damping force.

The frequency of the external force which makes the body to vibrate/ oscillate is called applied frequency (Forced frequency).

If the applied frequency matches with the natural frequency of the body then the amplitude of a body becomes maximum which leads to Resonance.

Example:

i) Motion of hammer in a calling bell.
ii) The vibrations of a ear drum caused by external sound.
iii) Periodic vibration of current in LCR circuit by an AC source.

Theory of Forced Oscillations:

Consider a body of mass m, executing oscillation/vibration due to external periodic force.

The External periodic force = F sin pt ———————— (1)

Where, p – angular frequency of external source. The direction of external force is opposite to the direction of resistive and restoring force.

The Restoring force acing on a body given by,

$$F_{restoring} = -k\,x \qquad \text{———————— (2)}$$

Where, k – force constant

x - displacement

The Resistive force is given by

$$F_{resitsive} = -r\frac{dx}{dt} \qquad \text{-------------------------- (3)}$$

Where, r – damping constant and it depends on nature of fluid.

Negative sign indicates Force, F is acts opposite to the displacement/ velocity.

∴ The resultant force/Net force acting on a body is

$$F_{Resultant} = F_{External} + F_{restoring} + F_{resitsive}$$

Substituting Eqn (1), (2) & (3), we get,

$$F_{Resultant} = F\,\text{sinpt} - k\,x - r\frac{dx}{dt} \qquad \text{———————— (4)}$$

From Newton's second law, the net/resultant force is given by

$$F_{Resultant} = m\frac{d^2x}{dt^2} \qquad \text{-------------------------- (5)}$$

Equating equation (5) and (4),

$$m\frac{d^2x}{dt^2} = F\,\text{sinpt} - k\,x - r\,\frac{dx}{dt}$$

$$\Rightarrow \quad m\frac{d^2x}{dt^2} + r\,\frac{dx}{dt} + k\,x = F\,\text{sinpt} \quad \text{------------------------- (6)}$$

Divide by 'm' on each term of equation (6),

Eqn (6) $\Rightarrow \quad \frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F}{m}\,\text{sinpt}$ ---------------------------- (7)

Let $\frac{r}{m} = 2b$ and $\frac{k}{m} = \omega^2$ (where ω – **natural frequency**)

Eqn (7) $\Rightarrow \quad \frac{d^2x}{dt^2} + 2b\,\frac{dx}{dt} + \omega^2\,x = \frac{F}{m}\,\text{sinpt}$ ------------------------- (8)

Equation (8) is the expression for Equation of motion for forced oscillation.

The solution of Eqn (8) is

$$x = a\sin(pt - \alpha) \quad \text{------------------------- (9)}$$

Where a and α **are unknown and to be found.**

Differentiating Eqn(9), twice with respect to x

Eqn (9) $\Rightarrow \quad \frac{dx}{dt} = a\cos(pt-\alpha)p$

And $\quad \frac{d^2x}{dt^2} = -a\sin(pt-\alpha)p^2$ ------------------------- (10)

Put Eqn (9), (10) in Eqn (8)

Eqn (8) $\Rightarrow \quad -a\sin(pt-\alpha)\,p^2 + 2b\,a\cos(pt-\alpha)\,p + \omega^2 a\sin(pt-\alpha)x = \frac{F}{m}\,\text{sinpt}$ -------- (11)

Let RHS of Eqn (11) can be written as, $\quad \frac{F}{m}[\sin(pt-\alpha)+\alpha]$ ------------------------- (12)

The equation is in the form sin(A+B) = sinA cosB + cosA sinB

$\therefore$Eqn (12) $\Rightarrow \quad \frac{F}{m}[\sin(pt-\alpha)\cos\alpha + \cos(pt-\alpha)\sin\alpha]$ ------------------------- (13)

Put Equation (13) in Eqn(11) , (replacing RHS)

Eqn (11) $\Rightarrow \quad -a\sin(pt-\alpha)\,p^2 + 2b\,a\cos(pt-\alpha)\,p + \omega^2 a\sin(pt-\alpha)x = \frac{F}{m}[\sin(pt-\alpha)\cos\alpha + \cos(pt-\alpha)\sin\alpha]$ ------- (14)

Equating the Coefficient of $\sin(pt-\alpha)$ on both sides of equation (14),

$$\Rightarrow \quad -ap^2 + a\omega^2 = \frac{F}{m}[\cos\alpha] \quad \text{------------------------- (15)}$$

Equating the Coefficient of $\cos(pt-\alpha)$ on both sides of equation (14),

$$\Rightarrow \quad 2abp = \frac{F}{m}[\sin\alpha] \quad \text{------------------------- (16)}$$

Squaring and adding Eqn(15) & Eqn (16)

$$\Rightarrow \quad [-ap^2 + a\omega^2]^2 + (2abp)^2 = \left(\frac{F}{m}\right)^2[\cos^2\alpha + \sin^2\alpha]$$

After simplifying, we get, $a^2[(\omega^2 - p^2)^2 + 4b^2p^2] = \left(\frac{F}{m}\right)^2$

$\therefore$ Amplitude of a Vibrating body, $a^2 = \sqrt{\frac{\left(\frac{F}{m}\right)^2}{(\omega^2-p^2)^2+4b^2p^2}}$

$\Rightarrow$ Amplitude, $a = \frac{(F/m)}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}}$ ---------------------- (17)

The Phase of a particle executing Forced Oscillation can be calculated using Eqn (15) & (16).

Eqn(16) ÷ Eqn(15) $\Rightarrow \frac{\frac{F}{m}[\sin\alpha]}{\frac{F}{m}[\cos\alpha]} = \frac{2abp}{-ap^2+a\omega^2}$

$\Rightarrow$ $\tan\alpha = \frac{2bp}{\omega^2-p^2}$

$\Rightarrow$ $\alpha = \tan^{-1}\left(\frac{2bp}{\omega^2-p^2}\right)$ ---------------------- (18)

Eqn(18) is the expression for Phase of a body executing Forced Oscillations.

As frequency of the applied force 'p' changed/varied with respect to natural frequency ω, then the following three cases can be discussed, which indicates variation of amplitude 'a' and phase 'α'.

Case (i): If $p \ll \omega$

For $p \ll \omega, \omega^2 - p^2 \cong \omega^2$

Eqn (17) $\Rightarrow$ Amplitude, $a = \frac{(F/m)}{\sqrt{(\omega^2)^2}} = \frac{(F/m)}{\omega^2 b} = \frac{F}{m\omega^2}$

$\Rightarrow$ $a = \frac{F}{m\omega^2}$ ---------------------- (19)

From Eqn(19), 'a' depends on F/m

∴ 'a' will become constant or minimum.

For $p \ll \omega$, Eqn (18) $\Rightarrow$ Phase, $\alpha = \tan^{-1}(0) = 0$

Hence displacement of the vibrating body and frequency of external force are in-phase.

Case (ii): If $p = \omega$

For $p = \omega, \omega^2 - p^2 = 0 \Rightarrow \omega = p$

Eqn (17) $\Rightarrow$ Amplitude, $a = \frac{(F/m)}{\sqrt{4b^2p^2}} = \frac{(F/m)}{2b\omega} = \frac{F}{2mb\omega}$

Since $b = \frac{r}{2m}$

On substituting, $a = \frac{F}{r\omega}$ ---------------------- (20)

From Eqn(20), 'a' varies inversely with damping force 'b'

∴ as damping force decrease, the amplitude of a vibrating body increases. This leads to **Resonance** i.e, maximum amplitude.

For $p = \omega$, Eqn (18) $\Rightarrow$ Phase, $\alpha = \tan^{-1}\left(\frac{2bw}{0}\right) = \tan^{-1}(\infty)$

$\alpha = \frac{\pi}{2}$

Hence displacement has a Phase lag (behind) of $\frac{\pi}{2}$ w.r.to phase of applied frequency.

Case (iii): If $p \gg \omega$

For $p \gg \omega, \omega^2 - p^2 \cong -p^2$

Eqn (17) $\Rightarrow$ Amplitude, $a = \frac{(F/m)}{\sqrt{p^4+4b^2p^2}}$ ---------------------- (21)

From Eqn(21), as p increases(becomes large) 'a' varies inversely with 'b & p'

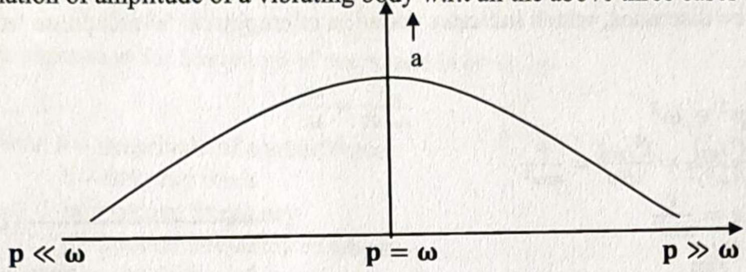
∴ The amplitude of a vibrating body decreases and becomes minimum.

For $p \gg \omega$, Eqn (18) ⟹ Phase, $\alpha = \tan^{-1}\left(\frac{-2bp}{p^2}\right) = \tan^{-1}\left(\frac{-2b}{p}\right)$

If b is small, then $\frac{2b}{p} \cong 0$ ∴ $\alpha = \tan^{-1}(0) = \pi$

As p is larger, the displacement of the vibrating body has thr Phase lag (behind) of π w.r.to phase of applied frequency.

The variation of amplitude of a vibrating body w.r.t all the above three cases is as shown in graph.



Resonance: It is the condition where the applied frequency matches with the natural frequency.

There are two conditions for resonance:

1. The applied frequency (p) should be equal to natural frequency (ω) of a body.
   i.e., $p = \omega$
2. The damping force 'b' related to the medium should be minimum.

Based on the above two condition, the amplitude of a body becomes maximum, given by

$$a_{max} = \frac{(F/m)}{2b\omega}$$

Example:

i) A radio receiver set tuned to the broadcast frequency of a transmitting station.

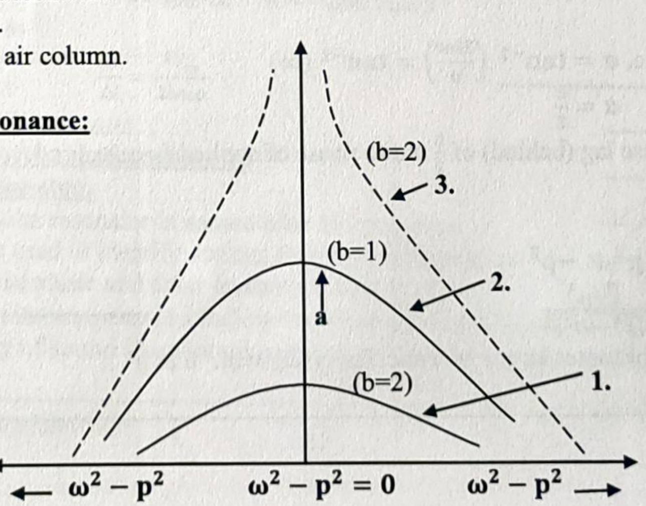
ii) Setting up of standing waves in the Melde's experiment.

iii) Helmholtz resonator.

iv) Sonometer.

v) Resonance air column.

Sharpness of resonance:



The graph indicates variation of amplitude v/s applied frequency.
In the graph,

1. Flat resonance (more damping)
2. Sharp resonance (less damping)
3. Infinite amplitude (zero damping)

The rate at which the change in amplitude occurs near the resonance depends on damping. As damping force increases, amplitude of a vibrating body decreases.

"The sharpness of resonance is the rate at which amplitude changes corresponding to the small change in the applied external frequency at the stage of resonance".

$\therefore$ Sharpness of resonance $= \frac{\text{Change in amlpitude}}{\text{change in external frequency}}$

$\Rightarrow$ Sharpness of resonance $= \frac{\Delta a}{\Delta f}$

The expression for Sharpness of resonance is given by

$$\frac{\Delta a}{\Delta f} = \frac{F/m}{2b\omega p}$$

Where, F – magnitude of applied force
b – damping force
ω = natural frequency
p – applied frequency of a body

Depends on the degree of damping, the sharpness of resonance is as shown in the graph. In the graph 'a' indicated amplitude of a vibrating body.

The expression for amplitude is given by

$$a = \frac{(F/m)}{\sqrt{(\omega^2-p^2)^2+4b^2p^2}} \quad \text{------------------------- (1)}$$

(i) At resonance, $\omega = p$

$\therefore$Eqn (1) $\Rightarrow$ $a_{max} = \frac{F/m}{2b\omega}$ ------------------------- (2)

(ii) Near to resonance, $\omega \approx p$

$\therefore$Eqn (1) $\Rightarrow$ $a = \frac{F/m}{2bp}$ ------------------------- (3)

$\therefore$ change in amplitude, $\Delta a = a_{max} - a = \frac{F/m}{2b\omega} - \frac{F/m}{2bp}$

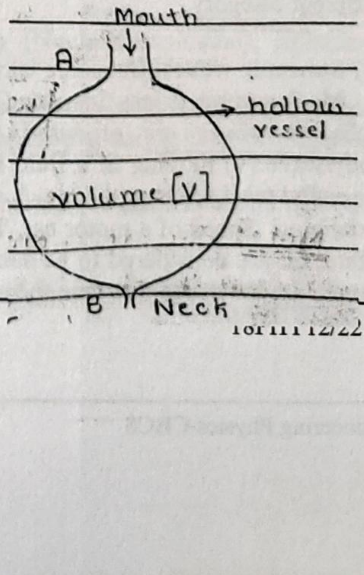
$\Rightarrow$ $\Delta a = \frac{F}{2bm}\left(\frac{1}{\omega} - \frac{1}{p}\right) = \frac{F}{2bm}\left(\frac{p-\omega}{p\omega}\right)$

Let $\Delta f = p - \omega$

Then we get, $\frac{\Delta a}{\Delta f} = \frac{F/m}{2b\omega p}$ ------------------------- (4)

Example for mechanical resonance:

Helmholtz Resonator:



Helmholtz resonator is named after Herman von-Helmholtz. It is used to identify various frequencies or musical pitches present in music and other complex sound.

Helmholtz resonator is a hallow vessel has two openings, one for the entry of sound (A-column) called mouth of large hole and other for the exit of sound called as neck of small/ narrow hole (B- column).

When a steam of air is running (blown) in and out throughout the vessel then at the mouth the air column moves inside the vessel in turn, the air inside the vessel compressed. To maintain inertia, the compressed air move upward to restore its volume.

Therefore the air inside the vessel makes compressions and rarefactions due to potential energy. The movement of air in to the vessel leads to kinetic energy. Hence some pressure difference is created and air particles start vibrating.

If the frequency of vibration ω (natural frequency) of air particle inside the vessel matches with the frequency of applied force then resonance takes place. Therefore maximum sound can be heard through neck due to maximum amplitude.

The square of the natural frequency varies inversely with the volume of resonator.

$\therefore \quad \omega^2 \alpha \frac{1}{V}$

$\Rightarrow \quad \omega^2 = \frac{k}{V}$

Where, k- constant of proportionality.

Note: *The expression for resonate frequency in a volume resonator is given by*

$$f = \frac{v}{2\pi}\sqrt{\frac{A}{V\,L}}$$

Where v – velocity of air column
A – area of mouth
L – length of mouth
V – volume of vessel.

SHOCK WAVES:

Mach number it is defined as the ratio of speed up object to the speed of the sound in the given medium.

Mach number, $\mathbf{M} = \frac{\textbf{speed of the object}}{\textbf{speed of sound in medium}} = \frac{v}{a}$

$\Rightarrow \quad \mathbf{M} = \frac{v}{a}$

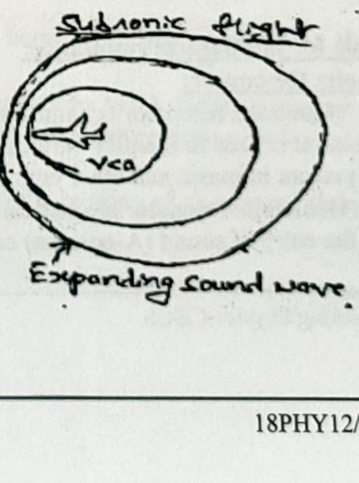
Mach number is the dimensionless number and it has no unit i.e. pure number

Depends on the Mach number the speed of the body moving in a medium are classified into different category.

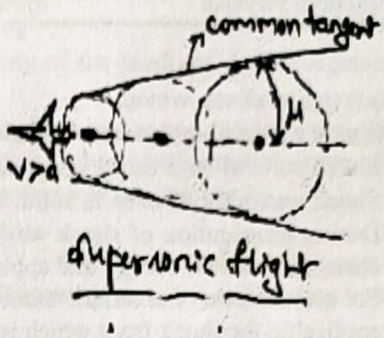
(1) Subsonic waves: Subsonic waves are those waves whose Mach number is less than one (M < 1).

Subsonic waves are produced when the speed of the body/wave (v) moving in a fluid is lesser than the speed of the sound (a), i.e. v < a.

Examples: Speed of a motor car, Train, Aero plane, flight of a bird, all are considered to be subsonic waves because they travel with the speed less than the speed of sound.



(2) **Super Sonic waves:** Super Sonic waves are those waves whose Mach number is greater than one (M > 1).



Super Sonic waves are mechanical waves which travel with speed greater than that of sound

Examples: Today's fighter plane can fly with supersonic speed, jet planes etc.

The amplitude of wave is high and it produces large changes in pressure and temperature of a medium during propagation.

A body with supersonic speed moves faster by piercing its own sound curtains, leaving behind a series of expanding sound waves with their centers displaced continuously along its trajectory.

Mach cone can be formed by drawing number of common tangents to the expanding sound waves generated by a supersonic flight.

Mach angle (μ): The angle made by the tangent with the axis of the cone is called Mach angle

The expression for Mach angle is given by

$$\mu = \sin^{-1}\left(\frac{1}{M}\right)$$

Where M - Mach number

(3) **Transonic waves:** Transonic waves are those waves whose Mach number is close to one. The range of transonic waves is lies between subsonic and supersonic waves. When a body changes its phase from subsonic to supersonic then a booming sound is heard which is known as sonic boom.

(4) **Hypersonic waves:** Hypersonic wave are those waves whose Mach number is greater than 5 i.e speed of object or sound is nearly 5 times greater than the speed of sound. Example scram jets have Mach number exceeding 5.

Shock waves:

Shock waves are produced by a sudden dissipation of mechanical energy in a medium enclosed in a small space. Any fluid that propagates at supersonic speed, give rise to shock waves. Examples: Seismic waves are the waves produced during earthquake which can travel with the speed of 2 km^{-1} and Lightning strikes during thunderstorms are the shock waves produced in nature.

Depends on the magnitude/ amplitude of shockwave they are generally classified in to two types

(a) **Strong shock wave:** Strong shock wave are those waves whose Mach number is greater than one (M>1) and they produce large changes in the pressure and temperature in the medium during their propagation in the medium

Examples: shock waves produced by explosion of bomb (Nuclear explosion), lightning Thunder, etc.

(b) **Weak shock wave:** Weak shock wave are those waves whose Mach number is close to one (M≈1) and they produce less changes in the pressure and temperature during their propagation in the medium

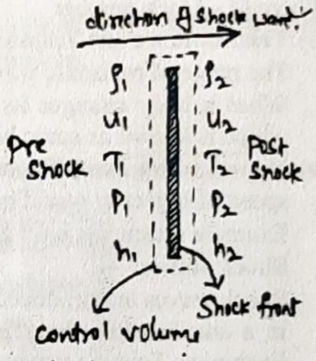
Examples: shock waves produced during burning of crackers, burning of an automobile tyre etc,

Properties of shock waves

1. Shock waves obey the laws of fluid dynamics.
2. Shock wave always travel in a medium with Mach number greater than one (M>1).
3. Shock waves travel even in solid, irrespective of fast energy dissipation.
4. During propagation of shock waves, the supersonic flow is decreased into subsonic flow with change in internal energy and applicable to adiabatic process.
5. For shockwaves we cannot associate the general wave properties, because shock waves are applicable for shock front which is similar like sound waves.
6. Shock waves are produced due to shock front which a region is held pressed in the space within the thickness of few micro meter and can capable to produce large instantaneous changes in the pressure and temperature.
7. The effects caused by shock waves, results in increase of internal energy-Entropy.
8. When shock wave turns around the edge/ convex corner they break up into large number of expanding waves with different Mach number. This process is known as supersonic expansion fan.

Control volume:

Control volume is a model used to analyze shock waves.
Control volume is a imaginary envelope consisting of shock front.
Control volume has two surfaces in one dimension.
The front surface is pre shock under and second surface is post shock.



Once the shockwave are produced, the physical parameters related to the medium changes. Therefore we consider density, flow velocity, change in internal energy, pressure and Enthalpy of of the medium.

Let $\rho_1, u_1, T_1, P_1, \& h_1$ be the density, velocity of fluid, temperature, pressure and enthalpy before creation of shockwave i.e., pre shock.
Let $\rho_2, u_2, T_2, P_2, \& h_2$ be the density, velocity of fluid, temperature, pleasure and enthalpy after the creation of shockwave i.e., post shock.

In control volume heat remains constant hence it is adiabatic process to study/understand shockwave, Rankine - Hugoniot relations are used.

Basic conservation laws:

(1) Law of conservation of mass: Mass is a matter associated with the body. The law of conservation of mass states that the total mass of the isolated system remains unchanged and is independent of any chemical/physical changes that take place within the system.
The relation for conservation of mass applicable to shockwave is given by

$$\rho_1, u_1 = \rho_2, u_2$$

Where ρ_1, ρ_2 are the density of the fluid before and after creation of shockwaves respectively and u_1, u_2 are the velocity of fluid before and after creation of shockwaves respectively.

(2) Law conservation of momentum: In an isolated system, total momentum of two interacting bodies remains conserved. It states that "In an isolated system when two bodies collides, the total momentum of the bodies before collision is equal to total momentum of the bodies after collision"

The relation for law of conservation of momentum applicable to shock wave is given by

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2$$

Where $\mathbf{P}, \boldsymbol{\rho}, \mathbf{u}$ indicates the Pressure, density, and fluid velocity of the medium. 1 & 2 represents before and after creation of shock waves respectively.

(3) Law of conservation of energy: It state that "energy can neither be created nor be destroyed, it can be transformed from one form to another, but total energy of the system remains constant"

The relation for law of conservation of energy applicable to shock wave is given by

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}$$

Where $\mathrm{h}, \mathbf{u}$ are the enthalpy and flow velocity of the medium. The prefixes 1 & 2 indicate for before and after creation of shock wave respectively.

Note:

(a) All the conservation laws are applicable to isolated and closed system.

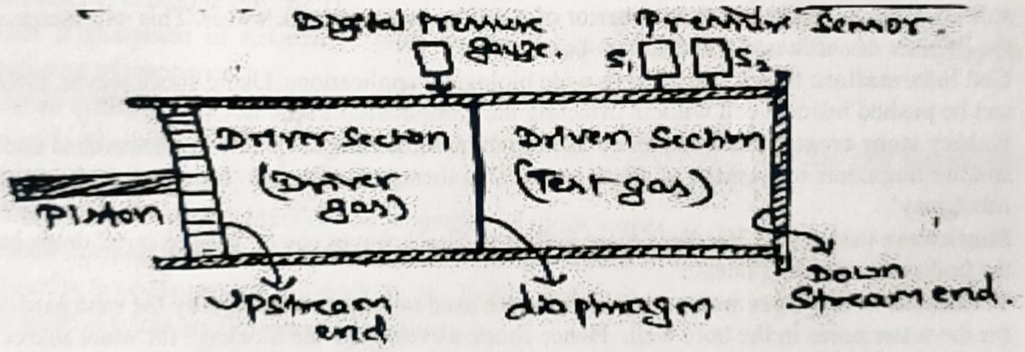
(b) Enthalpy = Internal Energy + (Pressure x Volumes).

Reddy Shock tube:

It is a hand operated shock tube capable of producing shock waves using human energy with Mach number greater than one (M > 1).

Reddy tube consists of long cylindrical tube with two sections separated by a diaphragm. One end is fitted by a piston and other end is either closed or open to surrounding.

<u>Construction and working of Reddy Shock tube:</u>



Construction: It consists of stainless steel cylindrical tube of length 1 m, and about 30 mm in diameter. The tube is divided into two parts of each length 50 cm by placing a diaphragm of thickness 0.1 mm, made up of paper or aluminum at the center. One part of the tube acts as driver tube on other part/section acts as driven tube. A movable frictionless piston is placed at the beginning of the driver section. Digital pressure gauze is mounted on the driver section close to the diaphragm.

Two Piezo electric sensors S_1 and S_2 are placed at a distance of 7 cm towards the closed end of the shock tube. A gas (Helium) at high pressure is filled at the driver section called driver gas and relatively lower pressure gas (Argon) is filled at driven section called test gas.

Working: The driver gas is composed by pushing the piston hard until the diaphragm breaks and expansion waves created. This shock wave (expansion waves) rushes in to driven section and pushes the driven gas towards the for-downstream end. This generated shock wave moves along the length of driver section, hence temperature and pressure of the test gas raises.

Reflected shock wave from the downstream end again compresses the test gas and increase its temperature and pressure. This state of high value of temperature and pressure is maintained at downstream and until the expansion wave are reflected from the upstream end from the driver tube to neutralize the compression partially.

The time period over which high value of temperature and pressure condition is sustained at downstream end is of the order of millisecond. This time period changes, depends on the dimension of the shock tube, nature of driver gas and test gas.

The Pressure sensor S_1 and S_2 are Piezo electrics transducers which senses the signal of primary shock waves produced by raise of pressure and temperature by reflected shock wave These signals are recorded in the digital cathode Ray oscilloscope (CRO). The oscilloscope of bandwidth one MHz or more is used for measurement of time based calculation for sustained state one milli second time interval.

If 'x' is the distance between two pressure sensor and 't' is the time taken by the shock wave to travel between two sensor, then velocity of the primary shockwave is given by

$$\mathbf{u_s} = \frac{x}{t} \text{ in ms}^{-1}$$

Applications of Shock waves:

1. **Wood preservation:** In wood samples like bamboo the chemical preservatives in the form of solutions can be pushed in to the interior of wood by passing shock waves. This will increase the lifetime of soft wood and ordinary bamboo.
2. **Cell information:** Shock waves have wide biological applications. Using shock waves, DNA can be pushed inside a cell without affecting the functionality of the DNA.
3. **Kidney stone treatment:** The kidney stones which blocks urinary tracts can be crushed into a smaller fragments by passage of shock wave. This therapy is called as 'Extra corporal lithotripsy'.
4. **Shockwave assisted in Needless drug delivery:** Shock waves can be used to inject drugs into the body without using needle.
5. **Treatment of dry Bore wells:** Shock waves are used to remove blockage by the sand particles for the water pores in the bore well. Hence shock waves clean the blockage for water source.
6. **Use in Pencil industry:** To soften the wood, the wood is placed in the liquid and shock wave is sent through. Thus liquid gets into the wood instantly and dry rapidly after removed from the liquid.
7. **Gas dynamics Study:** Shock wave can be used in the study of high temperature gas dynamics because, pressure and temperature of a fluid increases rapidly due to propagation of shock waves.

QUESTION BANK
MODULE 1 –OSCILLATIONS AND WAVES

1. Define simple harmonic motion. Derive an equation of motion for SHM & mention its solution.
2. Give the characteristics of simple harmonic motion.
3. Explain the theory of vertical vibrations of mass spring system. Mention the expression for time period & frequency. OR Describe the force constant for a mass suspended to a spring. Give the physical significance of force constant.
4. Derive an expression for equivalent force constant for 2 springs in series .Mention the expression for period of its oscillations.
5. Derive an expression for equivalent force constant for 2 springs in parallel. Mention the expression for period of its oscillations?
6. Explain how complex notation is represented.
7. Explain the phasor representation of SHM with examples.
8. What are damped oscillations? Give the theory of damped oscillations. Discuss the case of undedamping, critical damping & overdamping.
9. What are forced oscillations? Obtain an expression for amplitude and phase of a body undergoing forced vibrations. Discuss the case of i) $P<<\omega$, ii) $P=\omega$ and iii) $P>>\omega$.
10. Distinguish between free, damped and forced oscillations with an example.
11. What is resonance? Give the condition for amplitude of resonance. Explain one illustration of resonance (Helmholtz resonator).
12. What is sharpness of resonance? Give its significance. Explain the effect of damping on sharpness of resonance.
13. Define quality factor. Give its significance.
14. What is Mach number? Distinguish between subsonic, supersonic, Transonic and hypersonic waves.
15. What are shock waves? Mention the properties of shock waves.
16. Explain control volume. Discuss the basics of conservation of mass, momentum and energy.
17. Describe the construction and working of Reddy shock tube.
18. Mention any five applications of shock waves.

Module 1
Oscillations and Waves

Numericals on Oscillations

1. A spring of stiffness factor 98 N/m is pulled through 20 cm. Find the restoring force and compute the mass which should be attached so as to stretch in spring by the same amount.

Data: k=98 N/m, x=20 cm, F=? , m=?

Solution: Restoring force F= kx

$$= 98 \times 20 \times 10^{-2} = \mathbf{19.5\ N}$$

WKT, F = kx = mg

$$m = \frac{F}{g} = \frac{19.5}{9.8} = \mathbf{2\ kg}$$

2. Find the frequency of oscillation of a free particle executing SHM of amplitude 0.35 m if the maximum velocity it can attain is 220 m/s.

Data: a= 0.35 m, v_{max}= 220 m/s, ν =?

Solution: Equation for free vibration is $x = a \sin \omega t$

Velocity is $v = \frac{dx}{dt} = a\,\omega \cos \omega t$

$$= a\,\omega \sqrt{1 - \sin^2\omega t}$$

$$= \omega \sqrt{a^2 - x^2} \text{ (since } \sin \omega t = \frac{x}{a})$$

The particle attains maximum velocity at equilibrium position when x=0.

Hence $v = \omega \sqrt{a^2 - 0} = \omega\, a$

Angular frequency $\omega = \frac{v_{max}}{a} = \frac{220}{0.35} = \mathbf{628.5\ rad/s}$

Frequency of oscillation $\nu = \frac{\omega}{2\pi} = \frac{628.5}{2\times3.14} = \mathbf{100\ Hz}$

3. Calculate the resonance frequency for a simple pendulum of length 1m.

Data: L=1m, ν =?

Solution: $T = 2\pi\sqrt{\frac{L}{g}} = 2 \times 3.14\sqrt{\frac{1}{9.8}} = \mathbf{2\ s}$

Frequency of oscillation $\nu = \frac{1}{T} = \frac{1}{2} = \mathbf{0.5\ Hz}$

This is the natural frequency also called as Resonance frequency of the spring.

4. Find the displacement at the end of 3 seconds and also the amplitude of oscillations of a body executing SHM in a straight line if its period is 10 seconds ,and if its velocity is 1 m/s , at the time 2 seconds after crossing the equilibrium position. Assume that there are no resistive forces.

Data: T=10 s, a=? When v_1=1 m/s at t_1=2 s, x=? At t_2=3 s, x=?

Solution: Angular frequency $\omega = \frac{2\pi}{T} = \frac{2\times3.14}{10} = 0.628$ rad/s

Displacement $x = a \sin \omega t$

Velocity is $v = \frac{dx}{dt} = a\,\omega \cos \omega t$

$$a = \frac{v}{\omega \cos \omega t} = \frac{1}{0.628\times\cos(0.628\times2)} = \mathbf{5.14\ m}$$

(Note: $\cos(0.628 \times 2)$ is evaluated by using radian mode in calculator)

Now displacement at t_2=3 s is, $x = a \sin \omega t$

$$= 5.14 \times \sin(0.628 \times 3) = \mathbf{4.88\ m}$$

5. Evaluate the resonance frequency of a spring of force constant 2467 N/m, carrying a mass of 100 gm.

Data: k= 2467 N/m, m=100 gm = 100 x 10^{-3}kg, ν =?Solution: resonance frequency = natural frequency = ω = $\sqrt{\frac{k}{m}}$ and also ω = 2πν

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\times 3.14}\sqrt{\frac{2467}{100\times10^{-3}}} = \mathbf{25\ Hz}$$

6. A mass of 0.5 kg hangs from a spring .If the mass is pulled downward and let go it executes SHM. Calculate the period if the same spring is stretched 16 cm by 0.4 kg mass.

Data: m=0.5 kg, T=? When x= 16 cm and m=0.4kg

Solution: At equilibrium mg= kx

$$k = \frac{mg}{x} = \frac{0.4\times9.8}{16\times10^{-2}} = 24.5\ N/m$$

Now for m=0.5 kg the time period is,

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\times3.14\sqrt{\frac{0.5}{24.5}} = \mathbf{0.897\ s}$$

7. A 20 gm oscillator with natural angular frequency 10 rad/sec is vibrating in damping medium. The damping force is proportional to the velocity of the vibrator. If the damping co efficient in 0.17 kg/sec, how does the oscillator decay.

Data: m=20 gm, damping co efficient r = 0.17 kg/sec, ω = 10 rad/sec

Solution: WKT damping force F_d α v or $\frac{dx}{dt}$

Hence, $F_d = r\frac{dx}{dt}$

Damping factor $b = \frac{r}{2m} = \frac{0.17}{2\times20\times10^{-3}} = 4.25$

$b^2 = 18.06$ And $\omega^2 = 100$ here $\omega^2 > b^2$ hence oscillations are under damped.

8. A mass of 0.5 kg attached to a spring to set to vibrate .The vibratory motion is represented by the equation $\frac{1}{2}\frac{d^2x}{dt^2} + 0.014\frac{dx}{dt} + 1.5x = 0$ calculate the damping constant and angular frequency.

Data: m=0.5 kg

Equation of motion for damped oscillations is represented by,

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + \omega^2 x = 0$$

$$\frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{\omega^2}{m}x = 0$$

Now the given equation becomes $\frac{d^2x}{dt^2} + 0.028\frac{dx}{dt} + 3x = 0$

Here, $\frac{r}{m} = 2b = 0.028$ and $r = \mathbf{0.014\ kg/sec}$

$\omega^2 = \sqrt{3} = \mathbf{1.73\ rad/s}$

9. A mass of 200 gm is attached to a spring of negligible mass and the system is set for vibrations .If the damping constant for the system is 0.016 kg/s, then what will be the time required for the vibration amplitude to decay to 1/e of its starting amplitude. Estimate the number of vibrations the body executes meanwhile, if the force constant is 126 N/m (Assume the damping to be small).

Data: m=200 gm =0.2 kg, r=0.016 kg/s, k=126 N/m.

t_1=? (Time required for the vibration amplitude to decay to 1/e of its initial value)

n=? (No of vibrations the body executes in time t)

Solution: Amplitude of damped vibrations is, $A = ae^{-\left(\frac{r}{2m}\right)t}$

$$\text{At } t=0,\ A = ae^{-\left(\frac{r}{2m}\right)0} = ae^{-0} = a$$

Let at t_1 the amplitude decay from the value A to the value $\frac{A}{e}$

Now, $\frac{A}{e} = ae^{-\left(\frac{r}{2m}\right)t_1}$

$\frac{A}{e} = Ae^{-\left(\frac{r}{2m}\right)t_1}$ (since a = A)

$e^{-1} = e^{-\left(\frac{r}{2m}\right)t_1}$

Since the bases are same, the powers could be equated.

$-1 = -\left(\frac{r}{2m}\right)t_1$

$t_1 = \frac{2m}{r} = \frac{2\times0.2}{0.016} = \mathbf{25\ s}$

Number of vibrations per second is given by $T = 2\pi\sqrt{\frac{m}{k}} = 2 \times 3.14\sqrt{\frac{0.2}{126}} = \mathbf{0.25\ s}$

Therefore No of vibrations in t_1= 25 s is, $\frac{25}{0.25} = \mathbf{100\ Hz}$

10. A tuning fork has a natural frequency of 512 Hz .A periodic force per unit mass of amplitude 5×10^{-2} N/kg acts on it .Calculate the maximum amplitude attained by the fork if the damping per unit mass is 2×10^{-2} rad/s.

Data: $\nu = 512$Hz, $\frac{F}{m} = 5 \times 10^{-2}$N/kg, $\frac{r}{m} = 2 \times 10^{-2}$rad/s , a_{max}=?

Solution: The system attains maximum amplitude when it vibrates at its natural frequency for which

Eqn is, $a_{max} = \frac{\frac{F}{m}}{2b\omega}$

Where, $b = \frac{r}{2m} = \frac{2\times10^{-2}}{2} = 0.01$

$\omega = 2\pi\nu = 2 \times 3.14 \times 512 = 3215.36$ rad/s

$a_{max} = \frac{\frac{F}{m}}{2b\omega} = \frac{5\times10^{-2}}{2\times0.01\times3215.36} = \mathbf{0.77mm}$

Exercise Problems

11. A mass of 5 kg is suspended from the free end of a spring. When set for vertical oscillations, the system executes 100 oscillations in 40 seconds. Calculate the force constant of the spring.

12. A free particle is executing SHM in a straight line. The maximum velocity it attains during any oscillation is 62.8m/s. Find the frequency of oscillation, if its amplitude is 0.5 m.

13. A free particle is executing SHM in a straight line with a period of 25 seconds, 5 seconds after it has crossed the equilibrium point, the velocity is found to be 0.7 m/s. Find the displacement at the end of 10 seconds, and also the amplitude of oscillation.

14. Calculate the peak amplitude of vibration of a system whose natural frequency is 1000 Hz when it oscillates in a resistive medium for which the value of damping/unit mass is 0.008 rad/s under the action of an external periodic force/unit mass of amplitude 5 N/kg, with tenable frequency.

15. A vibrating system of natural frequency 500 cycles/second, is forced to vibrate with a periodic force/unit mass of amplitude 100×10^{-5} N/kg in the presence of a damping / unit mass of 0.01×10^{-3} rad/s . Calculate the maximum amplitude of vibration of the system.

Numericals on Shock Waves

1. The distance between two pressure sensors in a shock tube is 100 mm. The time taken by a shock wave to travel this distance is 200 μs. If the velocity of sound under the same conditions is 340 m/s, find the Mach number of the shock wave.

Data: x=100mm, t= 200 μs, M=?

Solution: Shock speed $u = \frac{x}{t}$

$$= \frac{100\times10^{-3}}{200\times10^{-6}} = 500\text{ms}^{-1}$$

Mach number $M = \frac{u}{a}$

$$= \frac{500}{340} = 1.47$$

2. Evaluate the speed of sound in helium gas at 350 K. Given, γ for Helium =1.667, R=2008$\text{Jkg}^{-1}\text{K}^{-1}$.

Data: γ=1.667, R=2008$\text{Jkg}^{-1}\text{K}^{-1}$, T=350 K, a=?

Solution: Speed of sound $a = \sqrt{\gamma RT} = \sqrt{1.667 \times 2008 \times 350} = 1082\ \text{m/s}$

Exercise Problems

3. The distance between two pressure sensors in a shock tube is 100 mm. The time taken by a shock wave to travel this distance is 100 μs. If the velocity of sound under the same conditions is 340 m/s, find the Mach number of the shock wave.

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Prepared by

Sl No	Faculty Name	Designation
1	Dr. Jagadeesha Gowda.G.V	Professor & HOD
2	Mrs. Shashikala.B.S	Asst.Prof
3	Mr. Gnanendra.D.S	Asst.Prof
4	Mr. Devaraja.C	Asst.Prof
5	Dr. Keshavamurthy.K	Asst.Prof
6	Ms. Namratha V	Asst.Prof
